Non-Linear Aspects of Black Hole Physics

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by

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To
The memory of my uncle
Anup Kumar Lala

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List of publications

1. "Ehrenfests scheme and thermodynamic geometry in Born-Infeld AdS black holes," **Arindam Lala**, D. Roychowdhury, Phys. Rev. D 86 (2012) 084027 [arXiv[hep-th:1111.5991]].

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This thesis is based on all the above mentioned papers whose reprints are attached at the end of the thesis.

Non-Linear Aspects of Black Hole Physics

Contents

Chapter 1

Introduction

1.1 Overview

We live in a universe surrounded by four types of forces: weak, strong, electromagnetic, and gravitational. While there is considerable amount of theoretical and experimental data that enables us to understand the nature of first three of these forces, there remains substantial ambiguity in understanding the true nature of the gravitational force[1]-[3]. Gravity has puzzled scientific community for centuries, and still it is a rather challenging avenue to make unprecedented expeditions.

In the beginning of the twentieth century, Albert Einstein proposed a new theory, the *Special Theory of Relativity*, which changed the age-old notion of space and time. Alike Newtonian mechanics, special relativity also depicts the structure of space-time, but the inconsistency of Newtonian mechanics with the Maxwell's electromagnetism was removed in the latter. Nevertheless, there is an apparent subtlety in the formulation of special relativity, it is a theory formulated entirely in inertial (i.e. non-accelerating) frames of references and it involves flat space-times having no curvature and hence there is no gravity.

The notion of gravity can easily be procured by shifting to the *General Theory of Relativity* (henceforth GTR). In this framework, gravity emerges quite naturally from the curvature of space-time which indeed implies that gravity is inherent to space-time. The core idea of GTR is encoded in the principle of equivalence which states that, the motion of freely falling particles are the same in a gravitational field and a uniformly accelerated frame in small enough regions of space-time. In this region, it is impossible to detect the existence of gravitational filed by means of local experiments[1]. Mathematically, the curvature of space-time is described by the metric tensor. The metric tensor encapsulates all the geometric and causal structure of space-time. In the classical GTR the dynamics of metric in the presence of matter fields is described by the celebrated Einstein's equations. These equations in fact relate the curvature of space-time with energy of matter fields[1].

The most fascinating thing about GTR is that it is not only a theory of gravity. There are more than that. This theory has been verified with success through experiments. On the other hand, with the thrill to understand Nature, many dazzling theories such as String Theory[4], the Standard Model of particle physics[2], etc. have been proposed. While each one of them deserves appreciation in their own right, there are close connections among GTR and these theories. As a matter of fact, classical GTR is an indispensable tool to explain many of the features of these theories. Despite the tremendous successes of GTR, a genuine quantum theory of gravity is still missing. This makes gravity more enigmatic than other forces of Nature, although with the advancement of our knowledge even more exciting features are expected to be disclosed in the future.

There are several useful applications of GTR such as, black holes, the early universe, gravitational waves, etc. While these fall into the regime of high energy physics and astrophysics, it is worth mentioning that GTR is being profoundly applied in modern technology, such as the *Global Positioning System (GPS)*, and many more. However, this thesis is solely devoted to the study of several crucial aspects of black hole physics which are one of the striking outcomes of the solutions of the Einstein's equations[1].

Black holes are usually formed from the gravitational collapse of dying stars. Due to the immense gravitational pull, no information can escape from within a black hole while information can enter into it. A black hole is characterized by at least one gravitational trapping surface, known as the *event horizon*, surrounding the region of intense gravitational field. Moreover, there exists a singularity of space-time within the event horizon, guaranteed by the *Hawking-Penrose singularity theorem*, which arises from the *geodesic incompleteness* of space-time[1]. Over the past several decades, different types of black hole solutions have been formulated. All these solutions are obtained from Einstein's equations which in turn are formulated from very specific actions through the use of the least action principle[1]. In accordance with no-hair theorems, there exist restricted number of stationary black hole solutions characterized by a small number of microscopic parameters such as, mass, charge, angular momentum, and no other free parameters. It should be emphasized that, all the facts discussed in this paragraph are in fact classical pictures of black holes.

It is to be noted that, the Einstein tensor, $\mathscr{G}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$, is the most general tensor in (3+1)-dimensions which has second derivatives in the metric tensor, $g_{\mu\nu}$[1]. In an attempt to generalize the Einstein tensor in higher dimensions (i.e., greater than 4), D. Lovelock proposed a generalized version of this tensor[5]. This is the most general second rank tensor with vanishing divergence and is constructed out of second derivatives of the metric tensor. Notably, the Lovelock tensor is non-linear in the Riemann tensor and differs from the Einstein tensor only if the space-time has more than four dimensions. Since it contains only second derivatives of the metric, the quantization of the linearized Lovelock theory is free of ghosts[5]-[7]. A few motivations for studying such non-linear theories can be mentioned as follows:

- The inherent property of the Einstein tensor and the Einstein-Hilbert Lagrangian of containing only upto second derivatives of the metric tensor leaves a scope to immediately generalize them by including non-linear, higher curvature terms in the Lagrangian while keeping the above mentioned property of these two quantities intact. As a matter of fact, this generalization can be performed upto any order, and the resulting Lagrangian is indeed a higher derivative Lagrangian[8].

- When the curvature of space-time is very large, e.g., in the very early universe with distances of the order of Planck's length, Einstein's theory is not well defined. In such circumstances non-linear gravity theories are conjectured to be appropriate for describing gravity[8]-[9].

- It is well known that, Einstein's theory, as a candidate for quantum theory of gravity, suffers from non-renormalizability problem. In an attempt to restore renormalizability, it was shown that the Einstein-Hilbert action should be supplemented with higher curvature terms[10]-[13]. However, there remains a caveat in this prescription, namely, the theory becomes non-unitary in the usual perturbation theory framework which forbids the higher curvature gravity theories to be considered as suitable candidates for quantum gravity[14].

- Indications for considering higher curvature terms also came from the small slope expansion of string theory models[15]. There it was argued that higher curvature terms, such as R^2 or $R_{\mu\nu}R^{\mu\nu}$, could appear in the effective Lagrangian if the slope expansion of the low energy field theory limit, upto terms with two space-time derivatives, were considered. Later on Zwiebach[16] and Zumino[17] pointed out the compatibility between the curvature squared terms and the absence of ghost particles in the low energy limits of string theories.

The Lovelock Lagrangian density containing higher curvature correction terms in D-dimensions is given by[5, 6, 7],

$$\mathscr{L} = \sum_{k=0}^{N} \alpha_k \lambda^{2(k-1)} \mathscr{L}_k, \tag{1.1}$$

where $N = \dfrac{D-2}{2}$ and $N = \dfrac{D-1}{2}$ for even D and odd D, respectively. α_k and λ are coupling constants with units of area and length, respectively. The constant α introduces a length scale $l_\alpha \sim \sqrt{\alpha}$ which represents a short distance scale where the Einstein gravity turns out to be corrected[6].[1] [2]

In the above equation

$$\mathscr{L}_k = \frac{1}{2^k}\sqrt{-g}\, \delta^{i_1 \cdots i_{2k}}_{j_1 \cdots j_{2k}}\, R^{j_1 j_2}_{i_1 i_2} \cdots R^{j_{2k-1} j_{2k}}_{i_{2k-1} i_{2k}} \tag{1.2}$$

is the Euler density of order $2k$, and all other quantities have usual meaning[19]. The term $\mathscr{L}_0$ is the usual cosmological term, $\mathscr{L}_1$ is the Einstein-Hilbert Lagrangian, $\mathscr{L}_2$ is the Lanczos Lagrangian, and so on.

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[1] From string theory perspective, α is proportional to the square of the string length scale, $\alpha' \sim \ell_s^2$[6].

[2] The higher curvature terms in the bulk are naturally generated from α' corrections on string *world-sheet*[18].

The Einstein's equations of GTR suggest that the presence of matter indeed plays an important role to generate curvature of space-time which in turn is manifested as the force of gravity. Mathematically, this leads to the addition of matter fields in the Einstein-Hilbert action[1]. With this addition, the Einstein tensor becomes proportional to the energy-momentum tensor ($\mathscr{T}_{\mu\nu}$) corresponding to the additional matter fields in the action: $\mathscr{G}_{\mu\nu} \propto \mathscr{T}_{\mu\nu}$. Generally speaking, the presence of matter fields is manifested as non-zero electromagnetic fields associated with the solutions which in turn act as a source of the energy-momentum tensor. In the context of gravity, one of the most important examples of coupling of matter fields with gravity that has been explored widely in the literature is the Maxwell electromagnetic fields. With this construction, the field equations are both Einstein's equations and Maxwell's equations where these two sets of equation are coupled together. This is admirable since the Maxwell electromagnetic field strength tensor ($F_{\mu\nu}$) enters Einstein's equation through $\mathscr{T}_{\mu\nu}$, while the metric tensor ($g_{\mu\nu}$) enters into the Maxwell's equations explicitly[1]. The gravity solution corresponding to this particular example is known as the Reissner-Nordström solution[1]. Interestingly, the form of the matter field is not restricted to the Maxwell field. There are variety of matter fields that have been studied in the context of gravity, such as the scalar fields (ϕ), tensor fields ($B_{\mu\nu}$), power Maxwell fields ($(F_{ab}F^{ab})^q,\ q > 1$), dilaton fields (φ), and so on. Along with higher derivative corrections to the gravity action (higher curvature corrections), it has been a crucial issue to study the effects of higher derivative corrections to the matter fields, especially of gauge fields in the usual Maxwell electrodynamics[20], in the gravity action. As a matter of fact, these corrections emerge in the low-energy effective action of string theory[21, 22].

For the past several years, constructing gravity theories in the presence of higher derivative corrections to the Maxwell action has been a popular research direction [23]-[33]. From these studies several intriguing features of gravity solutions have been emerged. Among them regular black hole solutions[23], validation of the zeroth and first law of black hole mechanics[24], interesting black hole and brane solutions[27]-[33], etc., may be mentioned. These higher derivative corrections may be viewed as non-linear corrections to the usual Maxwell electrodynamics. Among these non-linear theories of electrodynamics, the Born-Infeld (BI) theory has earned repeated attention for the past few decades. The non-linear BI theory was proposed in order to remove the infinite self-energy associated with a point-like particle in the Maxwell electrodynamics[20]. However, due to the emergence of quantum electrodynamics (QED), which could remove this divergence in self-energy in an effective way by the method of renormalization[34], the importance of the BI theory was lessened. But, since the discovery of the string theory and the D-brane theory it has been revived significantly[21, 22, 25, 35]. The regularity of electric field in the BI theory leads to non-singular solutions of the field equations[23]. In addition, BI theory is the only non-linear theory of electrodynamics with sensible weak field limit[22]. Also, it remains invariant under electromagnetic duality[25]-[26]. Among other non-linear theories of electrodynamics, BI theory can be distinguished by some unique properties like, absence of shock waves and birefringence phenomena[36]. Perhaps the most interesting and elegant regime for the application of the BI theory is the string theory. The BI theory effectively describes the low-energy behavior of the D-branes, which are nonperturbative solitonic objects in string theory[22]. Importantly, the dynamics of electromagnetic fields on D-branes are governed by the BI Lagrangian[35]. In string theory open strings end on D-branes and the end points are considered as point charges. These end points must not have infinite self-energy for a physically sensible theory making higher derivative BI corrections obvious[35]. Since, string theory requires the inclusion of gravity for its consistency, it is quite obvious to connect non-linear theories of electrodynamics with gravity.

It is to be noted that, two new types of Born-Infeld-like NEDs have been proposed recently, namely the exponential non-linear electrodynamics (ENE) and the logarithmic non-linear electrodynamics (LNE), in the context of static charged asymptotic black holes[37, 38, 39]. The matter actions with ENE and LNE also yield higher derivative corrections to the usual Maxwell action. Moreover, they retain several important properties of the Born-Infeld electrodynamics[38]-[40].

Apart from being an important ingredient for describing several aspects of theoretical physics, the non-linear theories of electrodynamics also bear significance from cosmological physics points of view. There are several examples in this regard. A few of them are the followings: (i) the early universe inflation may be explained with non-linear electrodynamics[41], (ii) the effects of non-linear electrodynamics become significant in the studies of pulsars, neutron stars, magnetars, and strange quark magneters [42, 43], (iii) inclusion of these theories into the photon dynamics leads to the dependence of gravitational red shift of strongly magnetized compact objects on the background magnetic field. This is due to the affect of these non-linear theories on the mass-radius ratio of the mentioned objects[43].

Although, gravity theories in $(3+1)$-dimensions are undoubtedly interesting and rich in structure, there are no reasons for not considering gravity theories in higher dimensions (i.e., more than four dimensions). First of all, from the mathematical structure of the Lovelock theory it is evident that in formulating generalized gravity theories (i.e., theories with higher curvature corrections) we must consider higher dimensions. On top of that, there are several other reasons which have accelerated the studies of gravity theories in higher dimensions, a few among them may be highlighted in the following points:

- Gravity is the manifestation of space-time geometry. Thus, from philosophical points of view, it is quite natural to ask what will be the nature of gravity if space-time possesses more than the conventional four dimensions.

- From the works of Kaluza and Klein it was evident that *small* extra-dimensions could be curled up within the perceived four dimensions in which we live[44].

- In string theory, which is a strong candidate for a quantum theory of gravity, higher dimensions emerges naturally as theoretical necessity[35]. Moreover, the study of microscopic theories of black holes leads to the consideration of higher dimensions. This is reassured by the fact that the microscopic counting of black hole entropy was first performed in a five dimensional black hole[45].

- Through the mathematical construction of "brane theory" it is observed that a $(3+1)$-dimensional brane, that describes our world, is in fact immersed in higher dimensional space[46].

- Black holes in higher dimensions possess many interesting features such as, possibility of rotation in several independent planes[47], presence of black saturns, black rings with horizon topology changing transitions[48], absence of spherical horizon topology[49], absence of stability in spinning black holes[50], and many others[3] which differ substantially from the four dimensional black holes. At the least, there is an intriguing possibility to check the validity of the known features of black holes, like uniqueness, laws of black hole mechanics, etc. in higher dimensions[51].

It is believed that, the study of GTR in higher dimensions may provide useful insight into the true nature of gravity which in turn may play an important role in the development of a quantum theory of gravity.

During the last four decades, considerable amount of research have been performed in the theory of black holes in the framework of GR. One of the main goals of these studies is to unravel the basic properties of black holes. Surprisingly, it has been observed that there is a fundamental relationship between the laws of black hole mechanics and the laws of ordinary thermodynamics. Subsequently, the successes of the thermodynamic descriptions of black holes in describing certain aspects of the quantum nature of strong gravitational fields have induced substantial amount of optimism to understand quantum nature of gravity.

The study of black hole thermodynamics was initiated in the early seventies, motivated by the similarity between the non-decreasing nature of black hole area and that of the conventional thermodynamic entropy. Starting with the seminal works of S. W. Hawking[52] and J. Bekenstein[53] series of works were pursued thereafter in this direction[54]-[58]. The close analogy between the laws of black hole mechanics and that of thermodynamics was first explicitly pointed out in Ref.[59] in the context of stationary, axisymmetric black hole solutions. There, based on some symmetry properties and energy conditions of the space-time, the following four laws were derived:

1. *Zeroth law:* "The surface gravity, κ of a black hole is constant over the event horizon."

2. *First law:* The change in black hole parameters (mass M, horizon area A, angular momentum J_H, and charge Q) are connected by the relation

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J_H - \Phi_H \delta Q \tag{1.3}$$

[3] See Ref.[51] for an interesting review.

where Ω_H and Φ_H are the angular velocity and electrostatic potential of the black hole defined on the horizon.

3. *Second law:* The area, A of the event horizon of a black hole is non-decreasing in time[52]:
$$\delta A \geq 0. \tag{1.4}$$

4. *Third law:* It is impossible to reduce κ to zero by any finite sequence of operations.

The mathematical form of the *first law* given above has a striking similarity with the first law of ordinary thermodynamics[60] which states that the difference in *energy (E)*, *entropy (S)* and any other state variables of two nearby equilibrium thermodynamic states are related as follow:
$$\delta E = T\delta S + \text{“relevant work terms”}. \tag{1.5}$$
A naive comparison between (1.3) and (1.5) leads to the following mapping between the black hole parameters and conventional thermodynamic variables:
$$E \leftrightarrow M, \quad T \leftrightarrow \frac{\kappa}{2\pi}, \quad S \leftrightarrow \frac{A}{4}.$$

Despite its success, this thermodynamical analogy of black holes was suspected. It continued to remain a mere mathematical construction without any physical meaning owing to the fact that classical black holes do not radiate at all. After all, the mapping $T\ dS \leftrightarrow \frac{\kappa dA}{8\pi}$ does not give us any clue how to separately normalize S/A or T/κ[1]. It was only after the discovery by S. W. Hawking that, due to quantum mechanical particle creation, black holes are capable of radiating all possible species of particles whose thermal spectrum resembles that of a perfect black body with a temperature
$$T = \frac{\kappa}{2\pi},$$
the quantities $A/4$ and $\kappa/2\pi$ could be interpreted as the *physical* entropy[4] and temperature of the black holes, respectively[55]. Thus, black holes can be considered as thermodynamic objects once quantum mechanical effects are taken into account. This so called *semi-classical approach* allows one to inspect into several thermodynamic aspects of black holes which include phase transition, critical phenomena, black hole evaporation, and so on.

Having identified black holes as thermodynamic objects with definite temperature and entropy, it is quite natural to look for whether they possess similar thermodynamic properties as conventional thermodynamic systems. Following this proposition, an extensive amount of research have been performed in this direction[69]-[123]. Among these, it is the black hole phase transition phenomena that has attracted researchers most over the years. There are some general motivations to study phase transitions in black holes, a few of which may be stated as follows:

[4]Entropy of black holes can also be calculated by following the methods prescribed in Refs. [61]-[68].

- It plays an important role to exploit thermodynamic phases of black holes as well as to understand the behaviors of the thermodynamic variables of interest. One can determine the orders of phase transitions and analyze the stability of black holes. Also, critical behavior of black holes as thermodynamic objects can be studied.

- It has been observed that, phase transitions in black holes may play an important role in order to understand the statistical origin of black hole entropy.

- As was shown by E. Witten, certain types of black hole phase transitions can be interpreted as *confinement-deconfinement transitions* of quark-gluon plasma in the context of gauge theories[80]. This opens up a promising direction to directly apply the theory of black holes in the studies of other fields.

- As will be discussed latter, phase transitions in black holes in bulk space-times are useful to study certain field theories on the boundary of the bulk via gauge/gravity dualities.[5]

From mid 70's till date, fascinated by the thermodynamic description of black holes, an enormous amount of effort has been given to study phase transitions in black holes. This study was started with the Schwarzschild black holes in asymptotically flat space-time[69, 70]. A Schwarzschild black hole is the unique, spherically symmetric, uncharged solution of vacuum Einstein equation[1]. By analytically continuing the black hole metric in the Euclidean sector, its Hawking temperature (T_H) can be easily determined[78] which comes out to be inversely proportional to the mass of the black hole (M). In studying the phase transitions, a unique scenario was imagined where the black hole was surrounded by thermal radiation. It was found that, in the flat space-time, the Schwarzschild black hole could be in thermal equilibrium with the thermal radiation at some temperature $T = T_H$. But this model has a serious flaw. The specific heat at constant volume is found to be negative which implies that the black hole is thermodynamically unstable. If we add additional mass to the black hole which is initially in thermal equilibrium with the thermal radiation, its temperature will go down and the rate of absorption will be more than the rate of emission. As a result the black hole will cool down and will continue to grow indefinitely. Alternatively, a small positive temperature fluctuation will decrease its mass and the temperature would be increased subsequently until all the mass radiates out. The above discussion points out the fact that, a canonical ensemble is ill-defined for a Schwarzschild black hole in asymptotically flat space-time. Latter on it was observed that due to the discontinuities in specific heats charged and rotating black holes might undergo phase transitions[71].

In an early attempt to obtain thermodynamic stability of Schwarzschild black holes, Hawking proposed a model in which a Schwarzschild black hole was contained in a box of finite volume and positive specific heat[57, 70]. It was observed that, the

[5] The present thesis discusses the first and the last of the above points in considerable details with several relevant examples.

black hole could be in stable thermal equilibrium with the radiation provided $E_R < \frac{M}{4}$, where E_R is the radiation energy of the box. This model, although unphysical, hid within its construction a clue to inspect into a more physical space-time geometry with boundary into which the thermodynamic stability of the Schwarzschild black hole could be found. Following this, gravitational theories were constructed in space-times having boundaries and hence finite spatial extension[72]-[76]. Interestingly, it was found that for a finite temperature at the spatial boundary the heat capacity of the black hole is positive. The space-times that are devoid of asymptotic flatness and have asymptotic curvature are those which admit a cosmological constant (Λ) in their geometry[1].

While for an asymptotically flat space-time $\Lambda = 0$, a space-time with nonvanishing Λ can be further classified into asymptotically de-Sitter (dS) and asymptotically anti de-Sitter (AdS) space-times. An asymptotically dS space-time admits $\Lambda > 0$ and has positive curvature. On the other hand, for an asymptotically AdS space-time $\Lambda < 0$ and it is negatively curved. Thermodynamics of black holes have been studied in both dS and AdS space-times. Interesting phase structures of black holes have been observed in both cases. On top of that, the subtlety that arises for black holes in asymptotically flat space-times has been successfully removed in considering these space-times[77, 78]. Let us give a brief overview of thermodynamics of black holes in AdS space-times in the following few sections.

Anti de-Sitter space has no natural temperature associated with it (similar to flat space but unlike dS space). This implies that the most symmetric vacuum state is not periodic in *imaginary time* although, it may be periodic in *real time*[78]. In an AdS space the gravitational potential relative to any origin increases as one moves away from the origin. Thus, the locally measured temperature of any thermal state decreases and the thermal radiation of massive particles remains confined near the black hole and they cannot escape to infinity. Although, zero rest mass particles can escape to infinity but the incoming and outgoing fluxes of such particles are equal at infinity. All these facts indicate that the AdS space behaves as a confining box, and there is no problem at all in considering a *canonical ensemble* description for black holes at any temperature[78].

Thermodynamics of black holes in AdS space-time differ substantially from their flat, non-AdS counterpart[78]-[123]. For example, unlike Schwarschild black holes in flat space, the temperature of Schwarzschild AdS black holes (henceforth SAdS) no longer decrease monotonically with mass. According to Ref. [78], there exits a phase transition, the *Hawking-Page transition* (henceforth HP), between the thermal AdS and SAdS in four dimensions. When the temperature of the thermal radiation is less than a minimum temperature T_0 black holes cannot exist and the space is dominated by radiation. When the temperature is greater than T_0 but less than a temperature T_1 there are two black hole solutions at equilibrium with the radiation. There exists a critical value of mass, M_0, for the black hole below which it has negative specific heat leading to instability of the lower mass black hole. Hence, the lower mass black hole may either decay into thermal radiation or to a higher mass black hole ($M > M_0$) with a positive heat capacity corresponding to a stable configuration. At

T_1 the free energy of the system changes sign and for $T_1 < T < T_2$ the black hole is thermodynamically favored over pure radiation. When $T > T_2$, all the thermal radiation collapse and the space is dominated by black holes. Motivated by the aforementioned elegance of AdS black holes, there has been a spate of papers in the literature[77]-[123] where several crucial aspects of these black holes have been studied over the years. In recent times, with the advancement of supersymmetric gauge and gravity theories and sting theory, especially of gauge/gravity correspondence, renewed attention has been paid in the study of black holes in AdS space[80]-[86].

Acceptance of the possible identification of black holes as thermodynamical systems has been evolved unanimously over the years. However, a completely different scheme to study phase transition and stability of black holes has been implemented very recently by R. Banerjee et. al.[115, 116] based on the familiar Ehrenfest's equations of standard thermodynamics[60]. The primary motivation that led to this investigation was to quantitatively describe the infinite discontinuities in the specific heats of black holes and to determine the order of thermodynamic phase transitions in black holes[115, 116]. It may be mentioned that, although, an infinite discontinuity in specific heat is the signature of a second order phase transition, it is not a sufficient condition to correctly identify the true order of the phase transition. In this circumstances, the Ehrenfest's equations of thermodynamics come into rescue[60]. For a second order phase transition, Ehrenfest's equations are simultaneously satisfied at the critical point(s). Even though the transition is not a genuine second order, its deviation from second order can be calculated via the Prigogine-Defay ratio, Π[60, 115]. Using this analytic scheme the nature of phase transitions in various conventional thermodynamic systems have been explored[124]-[128]. Not surprisingly, in applying the Ehrenfest's scheme in black hole thermodynamics all the relevant thermodynamic variables have been appropriately tailored in accordance with the laws of black hole mechanics mentioned earlier[59]. Using this analytic scheme, it has been possible to correctly classify the phase transitions in black holes[115]-[120]. Following this line of analysis, phase transition and stability of Kerr-AdS black[116] holes and charged Reissner-Nordström-AdS (henceforth RNAdS) have been discussed in Refs.[117]-[120].

Although, the Ehrenfest's scheme for black hole thermodynamics has been proven to be successful in clarifying the long-standing debates regarding the true order of phase transition, it consisted of a shortcoming. In Refs.[115]-[117], the authors computed both the Ehrenfest's equations close to the critical point(s) by means of numerical techniques. A true analytic method to check the validity of the Ehrenfest's equations at the critical point(s) was lacking. In an attempt to achieve this goal, an analytic technique has been developed subsequently, and based on this method we have been able to analytically calculate the mentioned equations exactly at the critical points, thereby determining the true order of black hole phase transitions which comes out to be of second order[119, 120].

In continuation of the preceding discussions, it is worth mentioning that, an alternative approach to describe thermodynamic systems in the framework of Riemannian geometry was proposed by G. Ruppeiner[129]-[135]. There it was shown that

thermodynamic systems could be represented by Riemannian manifolds once the theory of statistical fluctuation was included in the axioms of thermodynamics. In fact, it has been observed that aspects of thermodynamics and statistical mechanics of the system is encoded in the state-space metric [136]-[149]. The Riemann curvature of the manifold, which is derived from the state-space metric coefficients [129]-[135],[148], is associated with the interaction of thermodynamic states of the system [133]. These metric coefficients are in turn calculated from the Hessian of the thermodynamic entropy of the system and gives the pair correlation function[130, 131]. On the other hand, the exponential of the Ruppeiner metric gives the probability distribution of fluctuations around the maximum entropy state[130]-[149]. For ordinary thermodynamic systems, the absolute value of the Ricci scalar, $|R|$, is proportional to the correlation volume ξ^d (where ξ is the correlation length and d is the spatial dimension of the system). In the similar spirit, $|R|$ is interpreted as the average number of correlated 'Planck areas' on the event horizon[133]. While vanishing R implies that these areas are fluctuating independent of each other, a diverging $|R|$ implies highly correlated pixels indicating a phase transition[133]. Moreover, these divergences in $|R|$ is associated with the divergences of heat capacities and hence with the change of phases of the black holes[133].
One of the intriguing features of the Ehrenfest's scheme of phase transitions in black holes is that it is very much consistent with the Ruppeiner's thermodynamic state-space approach. It is reassured by the fact that the critical points of phase transition, where specific heats of black holes diverge, obtained in the previous approach, are exactly the point(s) where $|R|$ diverges, which is a characterizing feature of the latter[120].

In the theory of phase transitions in conventional thermodynamic systems, it is a common practice to study the behavior of the given system in the neighborhood of the critical point(s). This behavior is marked by the fact that various physical quantities that characterize the system exhibit singularities at the critical point(s). In fact, the divergence of the correlation length (ξ) of the system is manifested as the divergences of these quantities. The precise motivation of the theory of critical phenomena is to express these singularities in the form of power laws, characterized by a set of indices, known as the static critical exponents, that determine the critical behavior of the given system quantitatively[150]. These exponents are different from the dynamic critical exponents which characterize the diverging correlation length for non-equilibrium thermodynamic systems. It is found that, as $\xi \to \infty$, correlations extend over the macroscopic distances in the system. As a result, two different systems with different microscopic structures can no longer be differentiated from each other leading to a universal behavior of the systems[150]. In a second order phase transition the critical exponents are found to be universal for they do not depend upon the microscopic interactions present in the systems. It must be noted that, these exponents are not independent of each other, they are related to each other by the static scaling laws[151, 152]. Since black holes behave as thermodynamic systems, it is quite natural to study the qualitative behavior of the black holes via the critical phenomena and determine the critical exponents associated with the thermodynamic quantities pertaining to the black holes.

An exquisite attempt to determine the critical exponents for charged rotating black holes was made in Refs.[153]-[156], and the validity of the static scaling laws was checked explicitly for the same[155]. Also, critical phenomena in Reissner-Nordström black hole in asymptotically flat space-time was studied in Ref[157]. Thereafter, the critical phenomena in black holes has been explored at length in the literature[158]-[169]. Particularly, considering the growing successes of the AdS/CFT correspondence, the study of the critical phenomena has been extended to include black holes in AdS space-time[146, 147, 163, 164],[165]-[169]. In fact, from these studies, it has been observed that the theory of critical phenomena may shed light on several aspects of the duality. Moreover, these studies conclude that some black holes (such as R-charge black holes) obey the theory of critical phenomena in condensed matter physics[164].

Recently, interests in the study of black holes in AdS space-time is continuously growing in the context of the AdS/CFT duality (also called gauge/gravity duality, holographic duality, gauge/string duality)[170]-[175]. In this duality, it has been *conjectured* that certain aspects of quantum field theories (henceforth QFT) living on the boundaries of bulk AdS space-times can be realized by studying properties of black holes in the bulk AdS. Expressing in a precise language, this duality relates QFT and gravity: the quantum physics of strongly correlated many body systems is dual to the classical gravity in one higher dimension. In the original formulation, this duality related a four-dimensional conformal field theory (henceforth CFT), a scale invariant QFT, with type IIB string theory in ten-dimensions[170]. The precise statement is

$$\begin{aligned} d=4,\ \mathscr{N}=4,\ SU(N)\ &\text{Super Yang-Mills (SYM) theory at large}\ N \\ &\equiv \text{Type IIB superstring theory in}\ AdS_5\times S^5. \end{aligned}$$

It should be mentioned that, QFTs at strong coupling are extremely difficult to compute. It has been observed that the conventional perturbative methods are no longer reliable in analyzing strongly coupled systems. On the other hand, for strongly coupled quantum systems new weakly coupled dynamical degrees of freedom (d.o.f) emerge and the collective properties of these systems are well described by the fields associated with these d.o.f. The holographic duality is precisely an example of such a duality where these emergent fields live in a space with one extra dimension and the dual theory is a gravity theory. Surprising enough, this additional 'holographic dimension' is the manifestation of the energy scale of the quantum system under consideration[175].

Actually, the presence of such a duality was conceptualized earlier. In 1974 G. 't Hooft showed that, in the large N limit of the $U(N)$ color gauge group, with g^2N fixed (g: $U(N)$ coupling constant, N: number of color charges), the gauge theory could be considered as a theory of strings[176].[6] Subsequently, it was found that the algebra of three dimensional AdS space could be identified as the Virasoro

[6] In fact, the large N corrections resemble the g_s corrections in perturbative string theory.

algebra[177]. Also, Witten proposed that the phase transitions between thermal AdS and AdS black holes might be realized as the confinement/deconfinement transitions of quark-gluon plasma in the context of boundary conformal gauge theories[80].

It is noteworthy that, there exists many such dualities among physical theories, such as, QFT/QFT duality, string/string duality etc. Nevertheless, the AdS/CFT duality, an example of QFT/string duality, is much more useful from practical viewpoint, since, in constructing a quantum theory of gravity this duality allows us to reduce it to the problem of constructing a QFT[178, 179].

Although, the AdS/CFT duality was formulated in order to obtain better understanding of strongly correlated gauge theories, it is no longer restricted to the same. For example, it has been applied to determine of conserved quantities associated with a given space-time. This method is commonly known as the *counterterm method*[180]-[182]. This is a well known technique which removes the divergences in the action and conserved quantities of the associated space-time. These divergences appear when one tries to add surface terms to the action in order to make it well-defined. The counterterm method was applied earlier for the computation of certain conserved quantities associated with Kerr, Kerr-AdS space-times, and in the case of Lovelock gravity, dimensionally continued gravity as well[180]-[182]. On top of that, it has been observed that several crucial aspects of certain non-gravitational, strongly interacting physical systems can also be explained by using this duality as a theoretical *tool*. Perhaps the most important and pioneering example, that has helped achieving consistency between string theory and experiment, is the 'almost correct' estimation of the shear viscosity/entropy ratio (η/s) of the strongly interacting quark-gluon plasma (QGP), observed in the high energy collision experiment at RHIC, by the theoretical calculations using AdS/CFT correspondence[183]-[185].[7] Spurred by this accomplishment, several condensed matter systems have been studied under the framework of this duality. To mention a few, the theory of Hall effect and Nernst effect have been theorized in the context of dyonic black holes with the help of the AdS/CFT correspondence[187, 188]. An elaborated discussion on the connection between condensed matter physics and AdS/CFT can be found in Ref.[189]. These analyses revealed the fact that certain interesting properties (such as, thermal and electrical conductivities, and other transport coefficients) of strongly coupled, non-gravitational systems could be obtained by perturbing the black hole in the AdS space[185]. This holographic conjecture also found to be useful to obtain a correspondence between fluid dynamics and gravitational physics[190]. On the other hand, the contemporary discovery of spontaneous symmetry breaking and phase transition in AdS black holes by S. Gubser[191, 192] brought into focus the possibility of applying the duality further to quantify several properties of non-gravitational systems via the AdS/CFT duality.

One of the non-gravitational, condensed matter phenomena that has been studied extensively under the framework of AdS/CFT duality for the last ten years is the theory of high-T_c superconductivity[193]-[197]. But, before providing details about

[7] However, this bound is violated once higher curvature Gauss-Bonnet corrections are taken into account[186].

these *holographic superconductors*, let us make a historical review on the phenomena of superconductivity. This will help us develop a clear idea in building up the models of holographic superconductors in our subsequent discussions.

In 1911, H. K. Onnes first discovered the phenomena of superconductivity, namely, the electrical resistivity of most metals (e.g. mercury) drops abruptly to zero below a certain critical temperature[198]. Afterwards, it was observed that these superconducting materials also possess uncommon properties like, the persistence current and the complete expulsion of external magnetic field thereby making the material completely diamagnetic (known as the Meissner effect)[199]. Based on the Maxwell's theory of electromagnetism, in 1935, London brothers attempted to explain this phenomena quantitatively. They showed that an external magnetic field indeed decays inside a superconductor[200].

Latter on in 1950, a phenomenological model of superconductivity was put forward by Ginzburg and Landau[201]. They explained it in terms of a second order phase transition where a complex scalar field, ϕ, was introduced as the phenomenological order parameter. By analyzing the contribution of ϕ to the free energy of the system they noted that for temperatures (T) below the critical value, T_c, the order parameter becomes non-vanishing and a superconducting phase appears, whereas, for $T > T_c$ the order parameter vanishes and the normal conducting phase reappears. Notably, from particle physics point of view, this is quite similar to the Higgs mechanism which is associated with the spontaneous $U(1)$ gauge symmetry breaking[34].

In search for a complete microscopic theory, Bardeen, Cooper and Schrieffer proposed a new theory — the much known *BCS theory* of superconductivity[202]. By explicit field theoretic computations they showed that two opposite spin-1/2 electrons can interact with the phonons (the quanta of lattice vibrations) in such a way so as to form charged boson known as the *Cooper pair*. Moreover, below a critical temperature T_c, there is a second order phase transition and these bosons form a charged superconducting condensate[202]. Subsequently, based on this mechanism, properties of conventional superconductors were addressed successfully[203]. But in 1986, a new class of superconducting material was discovered with higher value of the critical temperature T_c. They were cuprate superconductors, and the superconductivity was along the CuO_2 plane[204]. However, the presence of these high-T_c superconductors was substantiated by the discovery of iron-pnictide superconductors in which the superconductivity is also found to be associated with two-dimensional planes[205]. Unfortunately, the mechanism of superconductivity in these high-T_c materials cannot be explained by the conventional BCS theory owing to the fact that, unlike the BCS theory, it involves strong coupling and the notion of Cooper pair may not even exist[195].

Till date, there is no satisfactory model that may explain the underlying mechanism of high-T_c superconductivity (although it is still believed that there might be a Cooper pair-like pairing mechanism driving the superconductivity). But, due to the presence of strongly interacting effective degrees of freedom in these materials AdS/CFT duality may bring upon some hope in this direction, since one of the aims of this dual gravity formulation is to explain strongly coupled field theories[170, 183].

The core idea of holographic superconductivity was originated from the work of S. Gubser[191, 192]. In this work, he showed that a charged, complex scalar field around an AdS black hole may explain the superconductivity holographically by the mechanism of spontaneous $U(1)$ gauge symmetry breaking. Based on this argument, he proposed that the Abelian-Higgs Lagrangian minimally coupled to the Einstein-Hilbert Lagrangian with a negative cosmological constant spontaneously breaks the local $U(1)$ gauge symmetry through the mechanism of a charged superconducting condensate formation close to the horizon of the black hole. This local symmetry breaking in the bulk corresponds to a global $U(1)$ symmetry breaking in the boundary field theory, in the dual AdS/CFT language. However, from the discussions in the preceding sections it is clear that, in order to obtain standard notion of superconducting phase transition holographically the condensate must vanish for $T > T_c$, whereas, it must form a superconducting condensate for $T < T_c$. The above mentioned holographic model seems to fulfill this criteria in the sense that, for $T > T_c$ we get the Reissner-Nordström black hole solution which resembles the conducting phase, whereas, for $T < T_c$ the non-vanishing scalar leads to a superconducting black hole solution. Motivated by the ingenuity of this approach a holographic model for superconductors was indeed built in 2008 in the pioneering research by Hartnoll *et. al.*[193]. This outstanding research ensured the close connection between gravity and condensed matter theory[193]-[197], [206]-[210].

Let us mention some genuine features of the holographic model of superconductors [193]-[197]. First of all, following the scheme of AdS/CFT duality, a holographic charged superconducting condensate in d-dimensions is realized by a $(d+1)$-dimensional bulk gravity theory in AdS space-time having a black hole. Secondly, in order to have the desired conductor/superconductor phase transition at a critical temperature (T_c) we must have a condensate at finite temperature. The Hawking temperature of the black hole in the bulk carries the notion of temperature in the boundary field theory. Thirdly, the second order phase transition at $T = T_c$, in the boundary field theory, resulting a transition from conducting to superconducting phase is dualized by a corresponding phase transition from a Reissner-Nordström black hole with no hair to a superconducting black hole with non-vanishing hair at $T = T_c$. At this point of discussion, the mechanism of condensate formation in the bulk AdS space-time is worth mentioning. The dual action for a d-dimensional holographic superconductor is given by[192, 193]

$$\mathscr{S} = \int d^{d+1}x \sqrt{-g} \left(R - 2\Lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - |\partial_\mu\phi - iqA_\mu\phi|^2 - m^2\phi^2\right) \tag{1.6}$$

where $g_{\mu\nu}$ is the AdS metric, R is the Ricci scalar, $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$, A_μ is the gauge field, and ϕ is the complex scalar with mass m and charge q. From (1.6) it is easy to note that, close to the horizon, g^{tt} becomes large and negative and hence the effective mass of the scalar, $m_e^2 = m^2 + q^2 g^{tt} A_t^2$, becomes sufficiently negative to destabilize the scalar field. As a consequence, the black hole becomes unstable to forming scalar hair[197]. Alternatively, when $qQ >> 1$ (Q: charge of the black hole) the strong electric field on the horizon pull pairs of oppositely charged particle out of the vacuum. Subsequently, particles having same charge as of the black hole repelled away but are forced to reside close to the black hole due to the confining property of the AdS space-time mentioned earlier in this chapter[78]. This ensemble of charged particles is precisely the holographic realization of the superconducting condensate[197].

Being familiar with the basic constructional details of the holographic superconductors, we may further mention some of its other properties that has been unveiled since the discovery of this holographic model[193]-[197],[206]-[210], [211]-[240]. These include: (i) The free energy of a black hole with scalar hair is always lower than that of a black hole without hair. The free energy difference between these two configurations always scales like $(T - T_c)^2$ near the critical point which indicates that a second order phase transition indeed occurs in going from conducting to superconducting phase. Moreover, this difference is removed when $T \rightarrow T_c$; (ii) The scalar field ψ, in the Abelian-Higgs part of the action (Eq. (1.6)), is put by hand and plays the role of order parameter for the phase transition in the bulk AdS. Then, according to the AdS/CFT dictionary, either of the coefficients of the boundary expansion of ψ can be taken as the order parameter ($\langle \mathscr{O} \rangle$) of phase transition in the boundary field theory, provided the mass of the scalar field satisfies the bound $-d^2/4 < m^2L^2 < -d^2/4 + 1$ (L : radius of the AdS space)[241, 242]. Near the critical point ($T = T_c$) it behaves as $\langle \mathscr{O} \rangle \sim (1 - T/T_c)^{1/2}$. This is a mean field behavior as predicted by the Ginzburg-Landau theory[201, 203]. In addition, the exponent 1/2 ensures that it is a second order phase transition; (iii) In the probe limit,[8] the real part of the conductivity, $Re[\sigma(\omega)]$, becomes a constant and independent of the perturbation frequency, ω, for $T > T_c$. On the contrary, for $T < T_c$ a frequency gap (ω_g) opens up at low-frequency, and subsequently the real part contains a delta function peak at $\omega = 0$ which corresponds to a pole in the imaginary part of the conductivity[193]. This suggests that there is an infinite dc conductivity for $T < T_c$. Moreover, the ratio $\omega_g/T_c \approx 8$, which is very close to the experimentally obtained result[243]; (iv) The non-linear gauge as well as gravity corrections (i.e., higher curvature corrections to Einstein gravity and higher derivative corrections to the Maxwell field) have effects on the properties of the holographic superconductors. Usually, with increasing non-linearity holographic superconducting condensate formation becomes difficult[213]-[240].

The expulsion of external magnetic field from inside a superconducting material is one of the distinct features of the phenomena of superconductivity[203]. This is known as the Meissner effect[199]. This property asserts that superconductors behave as perfect diamagnetic materials. It is also observed that, above a certain critical value of the external magnetic field the superconductivity is destroyed and the sample reverts back to the normal state. Based on this behavior, superconduc-

[8]In the probe limit, gravity and matter decouple and the back-reaction of the gauge and the scalar fields can be neglected in the neutral black hole background. This limit is achieved by rescaling the matter fields in (1.6) by the charge of the scalar field (q) and taking the limit $q \rightarrow \infty$. This simplifies the problem without hindering physical properties of the system.

tors are classified into two categories, namely, type I and type II[203]. For a type I superconductor, at $T < T_c$, there exists an upper critical magnetic field $B_c(T)$ below which no magnetic flux lines penetrate the sample. But, when the applied magnetic field $B > B_c$ the superconductivity is destroyed and the flux penetrates perfectly. On the other hand, for a type II superconductor, below the critical temperature, there exists two critical magnetic fields, $B_{c1}(T)$ and $B_{c2}(T)$ with $B_{c2} > B_{c1}$. When the external applied field is below $B_{c1}(T)$ no flux can penetrate the sample. Whereas, for $B > B_{c2} > B_{c1}$ superconductivity is completely destroyed. In the domain $B_{c1} < B < B_{c2}$ the magnetic flux partially penetrates the sample and it develops a microscopic structure consisting of both normal and superconducting regions known as the mixed state. It is in this region the Abrikosov vortex lattice is formed[244].

Motivated by the ability of the holographic duality in describing several known features of superconductors, responses of holographic superconductors in the presence of external magnetic fields have been studied at depth in the literature[230, 234, 239, 240],[245]-[258]. Based on these studies, it has been observed that these are indeed type II superconductors, and there is an upper critical value of the magnetic field above which the superconducting phase ceases to exist. Nevertheless, there is a notable difference between these models and real-life superconductors regarding the Meissner effect. From free-energy calculations it can be shown that there is no way an external magnetic field can be repelled from within a holographic superconductor. Any finite magnetic field always penetrates the 'sample'. This happens due to the consideration of the probe limit[194]. As a result, the condensates adjust themselves in such a way that they only fill a finite strip in the plane which reduces the total magnetic field passing through it[245]. However, the hope is that a gauged version of this holographic model may be able to expel magnetic fields in the usual way thereby exhibiting Meissner effect[194].

Based on these approaches, studies in magnetic response of holographic superconductors have been further extended. For example, vortex and droplet solutions have been constructed in the holographic superconductors[245]-[249]. Also, the superconducting coherence length (ξ) has been computed which is in agreement with the result of the Ginzburg-Landau theory[249].

Whenever one applies AdS/CFT duality to analyze strongly interacting field theories, it is implied that the boundary field theory is relativistic in which the Lorentz symmetry as well as the scale invariance is well preserved. This is manifested through the isotropic scaling symmetry of the system under consideration: $t \to \lambda t$, $\vec{x} \to \lambda \vec{x}$. But, it must be kept in mind while replicating condensed matter systems holographically that, most of these systems are non-relativistic in nature where Lorentz symmetry is broken explicitly[259, 260]. The behavior of these systems are governed by Lifshitz-like fixed points. These fixed points are characterized by the anisotropic scaling symmetry $t \to \lambda^z t$, $\vec{x} \to \lambda \vec{x}$. The exponent z is called the "dynamic critical exponent" and it measures the degree of anisotropy between space and time[259]. In order to describe these systems holographically, the standard prescriptions of AdS/CFT duality need to be modified due to the non-relativistic nature of the systems[261]-[266]. As a matter of fact, these dual prescriptions provide gravity duals for systems which are realized by non-relativistic CFTs (henceforth NRCFT). Along with this development, black hole solutions in the Lifshitz space-time has also been obtained[267]-[269] which may play an important role to understand finite temperature field theories in this non-relativistic framework.

There are some serious theoretical motivations for studying these systems. For example, Lifshitz-like fixed points appear in some strongly correlated systems (anisotropic sine-Gordon model[270], finite temperature multicritical points in the phase diagrams of some systems[271], the Rokhsar-Kivelson dimer model[272], some lattice models of strongly correlated electrons[273] etc.). The correlation functions in these systems exhibit ultralocality in space at finite temperature and fixed time which may play central role in explaining certain experimental observations[274]. Among other non-relativistic systems, special importance has been given for systems having Schrödinger symmetry ($z = 2$). Gravity dual for these systems have been proposed in Refs.[262, 263]. These studies are motivated mainly by experiments with fermions at unitarity and nucleon scattering experiments[275]. In this sense, study of non-relativistic field theories using gauge/gravity duality has much more utility in understanding real-life phenomena than its relativistic counterpart.

Over the past few years, motivated by the aforementioned facts, a series of work have been undertaken in order to understand several crucial aspects of holographic superconductors with Lifshitz scaling[276]-[290]. These studies have revealed the effects of anisotropy on characterizing properties of the superconductors. Also, effects of magnetic field on holographic Lifshitz superconductors have been studied, and alike AdS/CFT superconductors, interesting vortex and droplet solutions have been obtained[284, 286, 287, 288].

In passing an interesting point must be added. In all these works gravity models which are dual to the superconductors have been constructed. These are phenomenological models in the true sense. These models just reproduce the known properties of the high-T_c superconductors that have been explored earlier either phenomenologically or by explicit experiments[198]-[205]. It is believed that by appropriately embedding the model into string theory the microscopic properties of the superconductors may be revealed[194], which in turn may open up a new window to study microscopic properties of strongly correlated field theories much of which are still incomprehensible.

1.2 Outline of the thesis

The present thesis is based on the works [120, 169, 231, 240, 287], and in the entire thesis we aim to account for several non-linear aspects of black holes. We introduce non-linear corrections to the usual gauge field and/or to the usual Einstein gravity. Thus in our work we discuss non-linear corrections to the Einstein-Maxwell gravity under the framework of General Relativity. These corrections can be treated as higher-derivative corrections to the parent gravity theory in the sense that they include higher derivatives of the gauge fields (e.g., Maxwell gauge fields A_μ) and of gravity field (metric $g_{\mu\nu}$). In Chapter 2 and Chapter 3, based on the works in Refs.[120],[169], we explore various thermodynamic aspects of gauge and/or gravity corrected black holes in anti-de Sitter (AdS) space-time – the Born-Infeld-AdS (BI-AdS) black hole, and the third order Lovelock-Born-Infeld-AdS (LBI-AdS) black hole, respectively. In these chapters we discuss the behaviors of relevant thermodynamic quantities, stability, and phase structure of the mentioned black holes. For example, we determine the nature of phase transition in the BI-AdS black hole along with computing thermodynamic quantities like, temperature, entropy, mass, specific heat, etc. in Chapter 2; and a detail analysis of the critical behavior of the LBI-AdS black hole is presented in Chapter 3 where the static scaling exponents are computed, and the validity of the scaling laws are checked. Interestingly, our analysis is based on ordinary thermodynamic approach[60] with suitable modifications of the thermodynamic laws in accordance with the laws of black hole mechanics[59].

We further extend our study by exhibiting the effects of the aforementioned non-linearities on the properties of holographic s-wave superconducting phase transitions which is an important outcome of the consistent efforts that has been paid to declassify strongly interacting condensed matter systems via the AdS/CFT correspondence with high-T_c superconductors as an useful example. These are discussed in Chapters 4, 5, and 6. In Chapter 4 and Chapter 5, based on Refs.[231, 240], the non-linear corrections to the gauge and/or gravity sector of the usual Einstein-Maxwell-AdS gravity is considered, and modifications of properties of the holographic superconductors (such as, the critical temperature, order parameter, etc.) due to their presence is derived in detail. On the other hand, based on Ref.[287], in Chapter 6 the effects of non-linear, anisotropic scaling of the space and time coordinates in a Lifshitz space-time is considered. The derived black hole is indeed an anisotropic Lifshitz black hole characterized by a *dynamical critical exponent* that measures the anisotropy, and effects of this exponent on the important properties of the holographic Lifshitz superconductors is computed.

The whole thesis contains 6 *chapters*, including this introductory part. Chapter-wise summary is given below.

Chapter 2: *Thermodynamic Phase Transition in Born-Infeld-AdS Black Holes*

In this chapter we present in details the thermodynamic phase transition in Born-Infeld-AdS black holes. In our analysis we use an anlytic scheme which is common in ordinary thermodynamics, namely, the *Ehrenfest's scheme*. In our analysis the critical point of phase transition is marked by the points of discontinuity of the specific heat. This methodology enables us to explain the phase transition in ample details – a transition from lower mass black hole with negative specific heat to a large mass black hole with positive specific heat. Moreover, the validity of the two Ehrenfest's equations determines that the phase transition is a second order transition. The validity of this scheme is vindicated by analyzing the phase transition in the framework of thermodynamic state space geometry approach where the Ruppeiner scalar curvature is found to be diverging at the critical points found in the Ehrenfest's scheme.

Chapter 3: *Critical Phenomena in Higher Curvature Charged AdS Black Holes*

In this chapter, we further extended our investigation of thermodynamics of AdS black holes in presence of non-linear higher derivative corrections by studying critical phenomena in third order Lovelock-Born-Infeld-AdS black holes. We include gauge as well as gravity corrections in our model and investigate the scaling behavior in the mentioned black hole near the critical point(s) in a canonical ensemble framework. We explicitly compute all the static critical exponents associated with various thermodynamic entities of interest (e.g., specific heat, entropy, isothermal compressibility, etc.) and found that all these exponents indeed satisfy static scaling laws near the critical point(s). We also check the static scaling hypothesis for this black hole and find its compatibility with thermodynamic scaling laws. We also qualitatively discuss the thermodynamic stability and phase structure of the black hole, and based on this we argue that there may be a continuous higher order phase transition that essentially leads to thermodynamic stability of the black hole.

Chapter 4: *Holographic s-wave Superconductors with Born-Infeld Correction*

In this chapter we investigate several interesting properties of holographic s-wave superconductors in the planar Schwarzschild-AdS background in the framework of non-linear Born-Infeld electrodynamics. By explicit analytic methods we compute the critical temperature and order parameter of holographic condensation. We elaborate on the role of next-to-leading order boundary expansion coefficient of the scalar field (that eventually condenses below the critical temperature) as the order parameter, and compute the corresponding variation of the same with temperature. In order to solve this non-trivial problem we explicitly use the Sturm-Liouville eigenvalue method[216], and compare our result with those existing in the literature[224]. We observe that the condensation formation becomes difficult as the value of the Born-Infeld parameter increases. We also find that the critical exponent associated with the order parameter is 1/2, which is consistence with the mean field theory confirming a second order superconducting phase transition.

Chapter 5: *Gauge and Gravity Corrections to Holographic Superconductors: A Comparative Survey*

In this chapter, we study the onset of holographic s-wave condensation in the (4+1)-dimensional planar GaussBonnet-AdS (GB-AdS) black hole background. Along with this higher curvature correction to the Einstein gravity, we introduce higher derivative gauge field corrections to the Maxwell electrodynamics – the exponential (ENE) and logarithmic (LNE) nonlinear corrections. These corrections may be treated as Born-Infeld-like corrections for they posses similar properties as of the latter. Working in the probe limit and performing explicit analytic computations, with and without magnetic field, we find that these higher order corrections indeed affect various quantities characterizing the holographic superconductors – the critical temperature for condensation decreases with increasing non-linearity, and the order parameter and critical value of the magnetic field increase with increasing non-linearity. We perform a comparative study of the two aforementioned non-linear electrodynamics and show that the exponential electrodynamics has stronger effects on the formation of the scalar hair. We observe that our results are consistence with those obtained numerically in Ref.[236].

Chapter 6: *Holographic Lifshitz Superconductors and Their Magnetic Response*

In this chapter, we focus our attention to study the response of a holographic s-wave Lifshitz superconductor under the influence of an external magnetic field. These holographic models actually belong to a class of strongly interacting non-relativistic field theory described by Lifshitz gravity in the holographic sense. After providing basic properties of these superconductors existing in the literature[283], we derive interesting *vortex* and *droplet* solutions for this model. We observe that the obtained solutions remain unaffected by the presence of anisotropic scaling of time and space in the theory securing the fact that they are independent of the nature of the field theory concerned. We also compute the critical parameters of the condensation and show that they depend on the dynamical critical exponent that measures the anisotropy of the theory.

Chapter 7: *Summary and Outlook*

In this final chapter, we summarize our results followed by a discussion regarding some possible future directions.

Chapter 2

Thermodynamic Phase Transition In Born-Infeld-AdS Black Holes

2.1 Overview

In the early seventies the seminal research of Hawking and Bekenstein provided a thrust in the study of black holes[52]-[58]. Their works, followed by several others, established the fact that a black hole behaves like a thermodynamic system. A striking addition in those studies was provided by the authors of Ref.[59] showing the strange similarity between the laws of black hole mechanics and that of ordinary thermodynamics. Thereafter substantial amount of efforts were given in order to understand the thermodynamic properties of black holes, such as, their phase structure, behaviors of the thermodynamic quantities, etc. In this regard, the works of Hut and Davies may be mentioned[69, 70]. Nevertheless, the instability of the black holes in their formulation appeared to be a serious problem. However, the issue was resolved with the discovery of phase transition phenomena in the Schwarzschild black hole in anti-de Sitter space-time[78]. Till date several attempts have been made in order to describe the phase transition phenomena in black holes[69]-[123]. All these works are basically based on the fact that at the critical point(s) of phase transition the specific heat of the black hole diverges. Inspite of all these attempts, the issue regarding the classification of the nature of the phase transition in black holes remains highly debatable and worthy of further investigations.

In this chapter, based on the approach of Refs.[115]-[120], we discuss the exquisite attempt that has been made recently to resolve the above issue by adopting the Ehrenfest's scheme of ordinary thermodynamics[60]. In usual thermodynamics it is a general practice to adopt the Ehrenfest's scheme in order to classify the order of phase transition [124]-[128]. This has the following two basic advantages: (i) it is simple and elegant, and (ii) it provides a unique way to classify the nature of the phase transition in ordinary thermodynamic systems; even if the phase transition is not a truly second order transition, we can determine the degree of its deviation by defining a new parameter called Prigogine-Defay ratio (Π)[125]. The extension of Ehrenfest's scheme to black hole thermodynamics seems to be quite natural since black holes in many respects behave as ordinary thermodynamic objects.

On the other hand, constructing gravity theories in presence of various higher derivative corrections to the usual Maxwell electrodynamics has always been a popular direction of research. Some important observations in this regard have already been discussed in Chapter 1. In addition to that, let us mention some properties of Einstein-Born-Infeld theory that will play a central role throughout this chapter. This theory admits various static black hole solutions that possess several significant qualitative features which is absent in ordinary Einstein-Maxwell gravity. A few of them are given by the following: (i) The metric solutions in the Einstein-Born-Infeld theory differs significantly from that of the Einstein-Maxwell theory, and this results noticeable difference in the causal structure of the two solutions in the two theories[27]; (ii) Depending on the value of the electric charge (Q) and the Born-Infeld coupling parameter (b) it is observed that a meaningful black hole solution exists only for $bQ \geq 0.5$. On the other hand, for $bQ < 0.5$ the corresponding extremal limit does not exist. This eventually puts a restriction on the parameter space of BI-AdS black holes. This is indeed an interesting feature which is absent in the usual Einstein-Maxwell theory[110]; (iii) Furthermore, one can note that for $bQ = 0.5$, which corresponds to the critical Born-Infeld-AdS (BI-AdS) case, there exists a new type of phase transition (HP3) which has identical thermodynamical features as observed during the phase transition phenomena in non-rotating BTZ black holes[110]. This is also an interesting observation that essentially leads to a remarkable thermodynamical analogy which does not hold in the corresponding Reissener-Nordstöm-AdS (RN-AdS) limit.

All the above mentioned facts provide enough motivation to carry out a further investigation regarding the thermodynamic behavior of BI-AdS black holes in the framework of standard thermodynamics. We adopt the Ehrenfest's scheme of usual thermodynamics in order to resolve a number of vexing issues regarding the phase transition phenomena in BI-AdS black holes. The critical points correspond to an infinite discontinuity in the specific heat (C_Φ), which indicates the onset of a continuous higher order transition. At this point it is worthwhile to mention that, an attempt to verify the Ehrenfest's equations for charged (RN-AdS) black holes was first initiated in [117]. There, based on numerical techniques, the authors had computed both the Ehrenfest's equations close to the critical point(s). However, due the presence of infinite divergences in various thermodynamic entities (like, heat capacity, etc.), as well as the lack of analytic techniques, at that time it was not possible to check the Ehrenfest's equations exactly at the critical point(s). In order to address the above mentioned issues, the present paper therefore aims to provide an analytic scheme in order to check the Ehrenfest's equations exactly at the critical point(s). Moreover, the present analysis has been generalized taking the particular example of BI-AdS black holes which is basically the non-linear generalization of RN-AdS black holes[110]. Our analysis shows that it is indeed a second order phase transition.

In continuation, we apply the widely explored state space geometry approach[129]-[135] to analyze the phase transition phenomena in BI-AdS black holes. Our analysis reveals that the scalar curvature (R) diverges exactly at the critical points where C_Φ diverges. This signifies the presence of a second order phase transition, thereby vindicating our earlier analysis based on Ehrenfest's scheme.

Before we proceed further, let us mention about the organization of the present chapter. The classification of phase transitions in ordinary thermodynamics has been stated briefly in Section 2.2. In Section 2.3 we have discussed thermodynamic phases of the BI black hole in AdS space after briefly mentioning the geometric structure of the black hole (Section 2.3.1) and its thermodynamic quantities (Section 2.3.2). Using Ehrenfest's scheme, the nature of the phase transition in the BI-AdS black hole has been discussed in Section 2.4. In Section 2.5 we analyze the phase transition using the (thermodynamic) state space geometry approach. Finally, we draw our conclusions in Section 2.6.

2.2 Phase transitions and their classifications in ordinary thermodynamics

The phenomena of phase transition is very common in thermodynamics and statistical mechanics. It is responsible for almost all the physical processes that we encounter, from the simplest water-vapor transition to the exotic superconducting phase transition[60]. But not all phase transitions are equivalent. They can be classified depending upon the behavior of certain thermodynamic quantities associated with the respective systems. For example, a first order phase transition involves latent heat whereas, in a second order transition there is no involvement of latent heat. The order of a phase transition is usually determined by observing the discontinuity of the lowest order of the differential coefficients of the Gibbs function along the line of transition[60]. Thus for a first order phase transition which involves latent heat, the Gibbs free energy ($\mathscr{G}$) is continuous across the line, but its first derivatives $(\partial\mathscr{G}/\partial T)_P$ and $(\partial\mathscr{G}/\partial P)_T$ which gives the entropy (s) and specific volume (v) of the system respectively, are discontinuous. (Here T and P are the temperature and pressure of the system, respectively.) Similarly, for a second order phase transition in which there is no latent heat and no change in volume, the second derivatives of $\mathscr{G}$, representing specific heat, expansion coefficient, and compressibility are discontinuous.

Any first order phase transition (or any transition that occurs at constant T and P) can be described by the *Clausius-Clapeyron Equation*[60] given below:

$$\frac{dP}{dT} = \frac{\triangle s}{\triangle v} \tag{2.1}$$

where the lhs represents the slope of the $P-T$ curve (*the coexistence curve*), $\triangle s$ and $\triangle v$ are the change in the entropy and volume of the constituents of the two phases, respectively.

The *Clausius-Clapeyron Equation* is indeed derived from the Gibbs free energy $\mathscr{G} = U - Ts + Pv$, where it is assumed that the first derivatives of $\mathscr{G}$ with respect to the intrinsic variables (P and T) are discontinuous and their specific values in two phases give the slope of the coexistence curve at the phase transition point.

Contrary to the first order transition, a second order transition is described by the following two equations known as the *Ehrenfest's equations*[60]:

$$\left(\frac{\partial P}{\partial T}\right)_S = \frac{1}{VT}\frac{C_{P_2}-C_{P_1}}{\beta_2-\beta_1} = \frac{\triangle C_P}{VT\triangle\beta}, \tag{2.2}$$

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{\beta_2-\beta_1}{\kappa_2-\kappa_1} = \frac{\triangle\beta}{\triangle\kappa}, \tag{2.3}$$

where C_P, β, and κ are specific heat at constant pressure, volume expansion coefficient, and isothermal compressibility of the system, respectively. At the point of a second order phase transition both of these equations are satisfied. This is called a second order transition since C_P, β, and κ are second derivatives of $\mathscr{G}$. Note that, the suffices 1 and 2 denote two distinct phases of the system.

This classification can be extended further, however, as the order of phase transition increases the discontinuity in the relevant thermodynamic quantities becomes progressively less important and it becomes less relevant to think of the transition from one phase to another[60].

2.3 Thermodynamic phases of the Born-Infeld AdS black hole

2.3.1 Geometric structure of the black hole

The Born-Infeld black holes are a new class of solutions of the Einstein's equations. These are in fact charged black holes which can be treated as non-trivial non-linear generalizations of the Reissner-Nordström black holes since the ordinary Maxwell Lagrangian which is coupled to the gravity action for the latter is replaced by non-linear Born-Infeld Lagrangian[20]. Our primary target in this chapter is to study thermodynamics and phase transitions of $(3+1)$-dimensional Born-Infeld black hole in AdS space-time (with AdS radius L) (henceforth BI-AdS). The action that leads to the mentioned solution is given by[110]

$$\mathscr{S} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\Big[R - 2\Lambda + 4\mathscr{L}(F)\Big] \tag{2.4}$$

where R is the Ricci scalar derived from the metric $g_{\mu\nu}$ having determinant g, $\Lambda = -\dfrac{3}{L^2}$ is the cosmological constant which is negative for our AdS space-time, and

$$\mathscr{L}(F) = b^2\left(1 - \sqrt{1 - \frac{2F}{b^2}}\right) \tag{2.5}$$

is the non-linear Born-Infeld Lagrangian. The parameter b in Eq. (2.5) is related to the string tension α' as $b = \frac{1}{2\pi\alpha'}$. Note that, in the limit $b \to \infty$ the action Eq. (2.4) reduces to the Reissner-Nordström action[1]. In our analysis we always choose the system of natural units $\hbar = G = c = 1$. In this system of units the black hole metric can be written as[110]

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2 \tag{2.6}$$

where the metric coefficient $f(r)$ is given by

$$f(r) = 1 - \frac{2M}{r} + r^2 + \frac{2b^2r^2}{3}\left(1 - \sqrt{1 + \frac{Q^2}{b^2r^4}}\right) + \frac{4Q^2}{3r^2}\mathscr{F}\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, \frac{-Q^2}{b^2r^4}\right). \tag{2.7}$$

In Eq. (2.7) $\mathscr{F}\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, \frac{-Q^2}{b^2r^4}\right)$ is the hypergeometric function[291]. The only non-zero component of the gauge field strength $F_{\mu\nu}$, which gives the electric field ($\mathscr{E}$) of the BI-AdS black hole, is given by

$$F_{tr} = \mathscr{E} = \frac{Q}{r^2\sqrt{1 + \frac{Q^2}{b^2r^4}}}. \tag{2.8}$$

In the subsequent analysis we may choose $Q \geqslant 0$ and $b \geqslant 0$ without any loss of generality[110]. At this point the following limiting cases are worth noting: *(i)* In the limit $Q \to 0$, the metric Eq. (2.7) reduces to that of the Schwarzschild-AdS black hole; *(ii)* For $b \to \infty$ and $Q \neq 0$, Eq. (2.7) turns into the metric coefficient of the Reissner-Nordström-AdS black hole. In addition to that, the BI-AdS black hole possesses two horizons $r_\pm$ alike the Reissner-Nordström-AdS black hole but unlike the Schwarzschild-AdS black hole. In this regard the Schwarzschild-AdS black hole should be disconnected to the BI-AdS black hole[110]. The mass of the black hole is defined by $f(r_+) = 0$, which yields

$$M(r_+, Q, b) = \frac{r_+}{2} + \frac{r_+^3}{2} + \frac{b^2r_+^3}{3}\left(1 - \sqrt{1 + \frac{Q^2}{b^2r_+^4}}\right) + \frac{2Q^2}{3r_+}\mathscr{F}\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, \frac{-Q^2}{b^2r^4}\right) \tag{2.9}$$

where r_+ is the radius of the outer horizon.

2.3.2 Thermodynamic variables of the black hole

Being familiar with the basic geometry of the BI-AdS black hole we proceed further to derive several thermodynamic quantities that are most relevant to unravel the phase structure of the black hole. In this regard we need to tailor the ordinary thermodynamic variables appropriately in order to apply them to black hole physics in accordance with the laws of black hole mechanics[59], namely, the energy (E) is replaced by the mass (M) (Eq. (2.9)), pressure (P) is replaced by the negative of the electrostatic potential difference ($-\Phi$) and volume (V) is replaced by charge (Q) of the black hole.

To begin with, let us calculate the Hawking temperature of the black hole. In order to achieve this we analytically continue the metric Eq. (2.6) to the Euclidean sector by setting $t \to i\tau$. The resulting metric then require to be regular at the horizon (r_+). Thus, we must identify $\tau \sim \tau + \zeta$, where ζ ($= 2\pi/\kappa$; κ: the surface gravity of the black hole) is the inverse of the Hawking temperature, in order to avoid conical singularity at the origin. Following this line of analysis the Hawking temperature for the BI-AdS black hole is obtained as

$$T = \frac{1}{4\pi}\left[\frac{1}{r_+} + 3r_+ + 2b^2 r_+\left(1 - \sqrt{1 + \frac{Q^2}{b^2 r_+^4}}\right)\right]. \tag{2.10}$$

From the first law of black hole thermodynamics we get, $dM = TdS + \Phi dQ$[59]. Using this we obtain the entropy of the black hole as,

$$S = \int_0^{r_+} \frac{1}{T}\left(\frac{\partial M}{\partial r}\right)_Q dr = \pi r_+^2. \tag{2.11}$$

From this expression we observe that the entropy of the BI-AdS black hole is proportional to the area of the outer horizon. So, the usual area law for black hole entropy[59] also holds for the BI-AdS black holes.
Substituting Eq. (2.11) into Eq. (2.10) we rewrite the Hawking temperature as[110],

$$T = \frac{1}{4\pi}\left[\sqrt{\frac{\pi}{S}} + 3\sqrt{\frac{S}{\pi}} + \frac{2b^2\sqrt{S}}{\sqrt{\pi}}\left(1 - \sqrt{1 + \frac{Q^2\pi^2}{b^2 S^2}}\right)\right]. \tag{2.12}$$

The above expression for the temperature as a function of entropy allows us to analyze the behavior of the temperature which is important for gaining information about the phase structure of the black hole.

From Fig. 2.1 we clearly observe there is a 'hump' and a 'dip' in the $T - S$ graph. Another interesting thing about the graph is that, it is continuous in S. This continuous behavior of T rules out the possibility of first order phase transition. In order to check whether there is a possibility of higher order phase transition, we compute the specific heat at constant potential (C_Φ) (which is analog of the specific heat at constant pressure (C_P) in usual thermodynamics).

The electrostatic potential difference between the black hole horizon and the infinity is defined by Ref. [100] as

$$\Phi = \frac{Q}{r_+}\mathscr{F}\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, \frac{-Q^2}{b^2 r_+^4}\right). \tag{2.13}$$

Using Eq. (2.11) we further express Φ as

$$\Phi = \frac{Q\sqrt{\pi}}{\sqrt{S}}\mathscr{F}\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, \frac{-Q^2\pi^2}{b^2 S^2}\right). \tag{2.14}$$

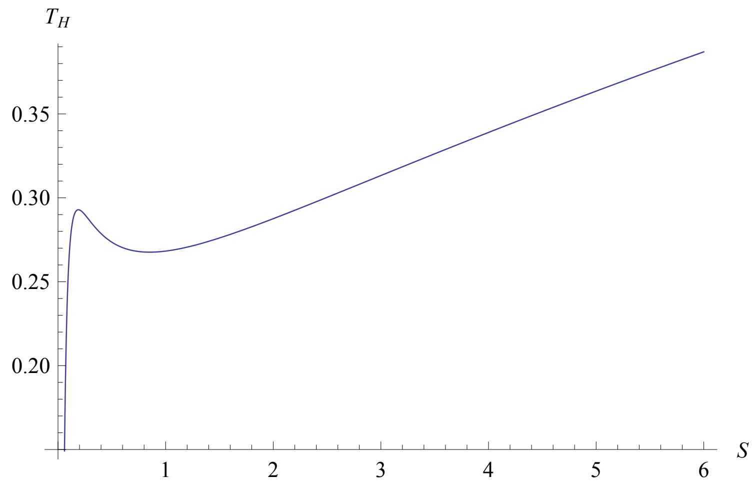

Figure 2.1: *Temperature (T) plot for BI-AdS black hole with respect to entropy (S) for fixed $Q = 0.13$ and $b = 10$.*

From the thermodynamical relation $T = T(S, Q)$, we find

$$\left(\frac{\partial T}{\partial S}\right)_{\Phi} = \left(\frac{\partial T}{\partial S}\right)_{Q} - \left(\frac{\partial T}{\partial Q}\right)_{S}\left(\frac{\partial \Phi}{\partial S}\right)_{Q}\left(\frac{\partial Q}{\partial \Phi}\right)_{S}, \tag{2.15}$$

where we have used the thermodynamic identity

$$\left(\frac{\partial Q}{\partial S}\right)_{\Phi}\left(\frac{\partial S}{\partial \Phi}\right)_{Q}\left(\frac{\partial \Phi}{\partial Q}\right)_{S} = -1. \tag{2.16}$$

Finally, using Eqs. (2.12), (2.14) and (2.15), the heat capacity C_Φ is expressed as,

$$C_\Phi = T\left(\frac{\partial S}{\partial T}\right)_{\Phi} = \frac{\mathscr{N}(Q,b,S)}{\mathscr{D}(Q,b,S)}, \tag{2.17}$$

where

$$\begin{aligned}\mathscr{N}(Q,b,S) = -2S&\left\{\pi + \left(3 - 2b^2\left(-1+\sqrt{1+\frac{Q^2\pi^2}{b^2S^2}}\right)\right)S\right\}\\ &\left\{b^2S^2 + \left(\pi^2Q^2 + b^2S^2\right)\mathscr{F}\left(\frac{3}{4},1,\frac{5}{4},\frac{-Q^2\pi^2}{b^2S^2}\right)\right\}\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}\mathscr{D}(Q,b,S) = b^2S^2&\left\{\pi + \left(-3 + 2b^2\left(-1+\sqrt{1+\frac{Q^2\pi^2}{b^2S^2}}\right)\right)S\right\}\\ &+ 2b^2S\left(-\pi Q + bS\right)\left(\pi Q + bS\right)\mathscr{F}\left(\frac{1}{4},\frac{1}{2},\frac{5}{4},\frac{-Q^2\pi^2}{b^2S^2}\right)\\ &+ \left(\pi - 3S - 2b^2S\right)\left(\pi^2Q^2 + b^2S^2\right)\mathscr{F}\left(\frac{3}{4},1,\frac{5}{4},\frac{-Q^2\pi^2}{b^2S^2}\right).\end{aligned} \tag{2.19}$$

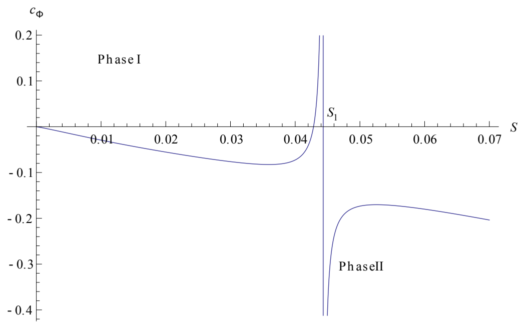

Figure 2.2: *Plot of specific heat (C_Φ) against entropy (S), at the first critical point (S_1), for fixed $Q = 0.13$ and $b = 10$.*

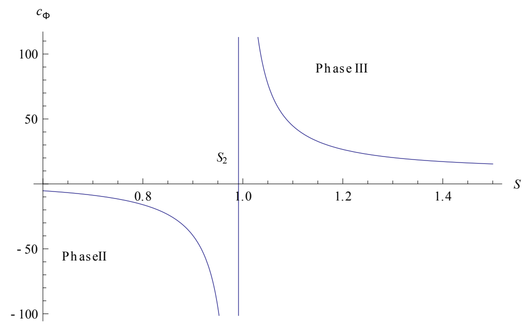

Figure 2.3: *Plot of specific heat (C_Φ) against entropy (S), at the second critical point (S_2), for fixed $Q = 0.13$ and $b = 10$.*

Let us first plot $C_\Phi - S$ graphs (Fig. 2.2 & Fig. 2.3) and make a qualitative analysis of the plots.

From these figures it is evident that the heat capacity (C_Φ) suffers discontinuities exactly at two points (S_1 and S_2) which correspond to the critical points for the phase transition in the BI-AdS black hole. Similar conclusion also follows from the $T - S$ plot (Fig. 2.1), where the 'hump' corresponds to S_1 and the 'dip' corresponds to S_2.

The graph of $C_\Phi - S$ shows that there are three phases of the black hole - Phase I ($0 < S < S_1$), Phase II ($S_1 < S < S_2$) and Phase III ($S > S_2$). Since the higher mass black hole possesses larger entropy/horizon radius, therefore at S_1 we encounter a phase transition from a smaller mass black hole (phase I) to an intermediate (higher mass) black hole (phase II). On the other hand, S_2 corresponds to the critical point for the phase transition from the intermediate black hole (phase II) to a larger mass black hole (phase III). Finally, Fig. 2.2 and Fig. 2.3 also let us conclude that the heat capacity (C_Φ) is positive for phase I and phase III, whereas it is negative for phase II. Therefore, phase I and phase III correspond to the thermodynamically stable phases ($C_\Phi > 0$), whereas phase II stands for a thermodynamically unstable phase ($C_\Phi < 0$).

As a next step, we calculate the volume expansion coefficient (β) and the isothermal compressibility (κ) of the black hole which will play a central role in the analytic calculations to be presented in the next section. In the case of the BI-AdS black hole these are defined as

$$\beta = \frac{1}{Q}\left(\frac{\partial Q}{\partial T}\right)_\Phi, \tag{2.20}$$

$$\kappa = \frac{1}{Q}\left(\frac{\partial Q}{\partial \Phi}\right)_T. \tag{2.21}$$

Using Eqs. (2.12) and (2.14), and considering the thermodynamic relation

$$\left(\frac{\partial Q}{\partial T}\right)_\Phi = -\left(\frac{\partial \Phi}{\partial S}\right)_Q \left(\frac{\partial Q}{\partial \Phi}\right)_S \left(\frac{\partial S}{\partial T}\right)_\Phi$$

we see that

$$\beta = \frac{-8b^2\pi^{\frac{3}{2}}S^{\frac{5}{2}}}{\mathscr{D}(Q,b,S)}, \tag{2.22}$$

where the denominator is identified as Eq. (2.19).

In order to calculate κ, we make use of the thermodynamic identity

$$\left(\frac{\partial Q}{\partial \Phi}\right)_T \left(\frac{\partial \Phi}{\partial T}\right)_Q \left(\frac{\partial T}{\partial Q}\right)_\Phi = -1. \tag{2.23}$$

Using Eq. (2.16) we obtain from Eq. (2.23)

$$\left(\frac{\partial Q}{\partial \Phi}\right)_T = \frac{\left(\frac{\partial T}{\partial S}\right)_Q \left(\frac{\partial Q}{\partial \Phi}\right)_S}{\left(\frac{\partial T}{\partial S}\right)_\Phi}. \tag{2.24}$$

Using Eqs. (2.12), (2.14), (2.15), and (2.24) we finally obtain

$$\kappa = \frac{\Psi(Q,b,S)}{\mathscr{D}(Q,b,S)} \tag{2.25}$$

where

$$\begin{aligned}\Psi(Q,b,S) = \frac{-2b^2S^{\frac{3}{2}}}{Q\pi^{\frac{1}{2}}} \Bigg[& 2\pi^2Q^2 - \pi S\sqrt{1+\frac{Q^2\pi^2}{b^2S^2}} \\ & + 2b^2\left(-1+\sqrt{1+\frac{Q^2\pi^2}{b^2S^2}}\right) + 3\sqrt{1+\frac{Q^2\pi^2}{b^2S^2}}\Bigg]\end{aligned} \tag{2.26}$$

and the denominator $\mathscr{D}(Q,b,S)$ is given by Eq. (2.19).

Note that, the denominators of both β and κ are indeed identical with that of C_Φ. This is manifested in the fact that both β and κ diverge exactly at the point(s) where C_Φ diverges. This is reassured by comparing the following plots (Figs.2.4, 2.5, 2.6, 2.7) for β and κ with Figs. 2.2, 2.3 showing the divergences in C_Φ.

From the above discussions it is evident that the phase transitions we encounter in BI-AdS black holes are indeed continuous higher order. In order to address it more specifically (i.e., whether it is a second order or any higher order transition) we adopt a specific scheme, known as *Ehrenfest's scheme* in standard thermodynamics, which will be the main topic of discussion in the next Section 2.4.

2.4 Study of phase transition using the Ehrenfest's scheme

In Fig. 2.2 and Fig. 2.3 we observed discontinuity in the heat capacity. However, discontinuity in the heat capacity does not always imply a second order phase transition, rather it suggests a continuous higher order transition in general. Ehrenfest's

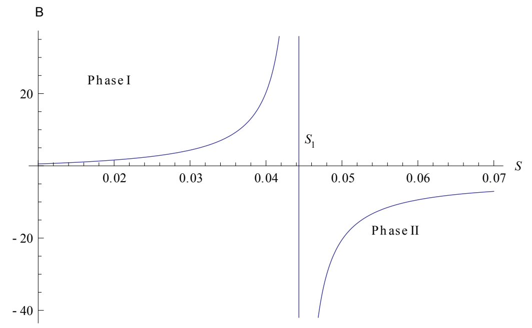

Figure 2.4: *Plot of volume expansion coefficient (β) against entropy (S), at the first critical point (S_1), for fixed $Q = 0.13$ and $b = 10$.*

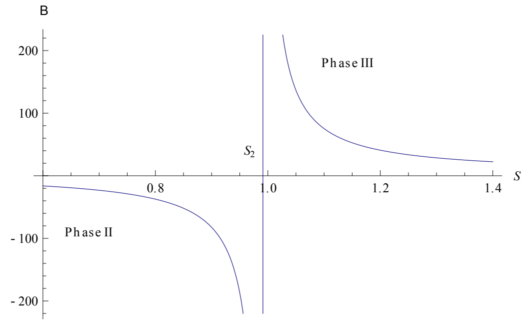

Figure 2.5: *Plot of volume expansion coefficient (β) against entropy (S), at the second critical point (S_2), for fixed $Q = 0.13$ and $b = 10$.*

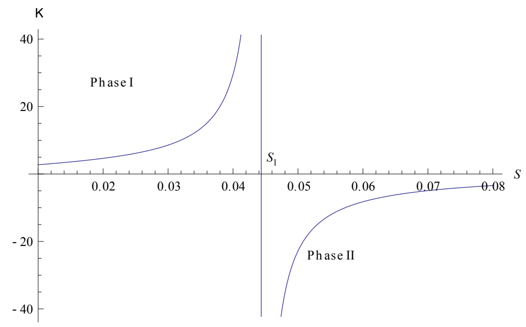

Figure 2.6: *Plot of isothermal compressibility (κ) against entropy (S), at the first critical point (S_1), for fixed $Q = 0.13$ and $b = 10$.*

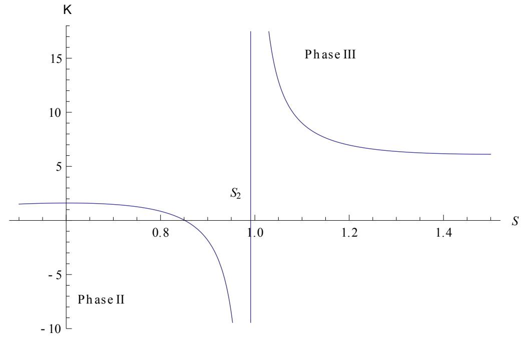

Figure 2.7: *Plot of isothermal compressibility (κ) against entropy (S), at the second critical point (S_2), for fixed $Q = 0.13$ and $b = 10$.*

equations play an important role in order to determine the nature of such higher order transitions for various conventional thermodynamical systems[124]-[128] as discussed in Section 2.2. This scheme can be applied in a simple and elegant way in standard thermodynamic systems. The nature of the corresponding phase transition can also be classified by applying this scheme. Moreover, even if a phase transition is not a genuine second order, we can determine the degree of its deviation by calculating the Prigogine-Defay (PD) ratio[124, 125, 126]. Here we apply a similar technique to classify the phase transition phenomena in (BI-AdS) black holes and check the validity of Ehrenfest's scheme for black holes. Thus, after suitable replacements of the thermodynamic variables, as demanded by the laws of black hole mechanics[59] (cf. Section 2.3.2), the modified Ehrenfest's equations (Eqs. (2.2), (2.3)) become[115]-[135]

$$-\left(\frac{\partial\Phi}{\partial T}\right)_S = \frac{1}{QT}\frac{C_{\Phi_2}-C_{\Phi_1}}{\beta_2-\beta_1} = \frac{\triangle C_\Phi}{QT\triangle\beta}, \tag{2.27}$$

$$-\left(\frac{\partial\Phi}{\partial T}\right)_Q = \frac{\beta_2-\beta_1}{\kappa_2-\kappa_1} = \frac{\triangle\beta}{\triangle\kappa}, \tag{2.28}$$

respectively.

In order to determine the order of the phase transition we analytically check the validity of the two Ehrenfest's equations (Eq. (2.27) & Eq. (2.28)) at the points of discontinuity S_i ($i = 1, 2$). Furthermore, for notational convenience, we denote the critical values for the temperature (T) and charge (Q) as T_i and Q_i, respectively.

Let us now calculate the lhs of the first Ehrenfest's Eq. (2.27), which may be written as

$$-\left[\left(\frac{\partial\Phi}{\partial T}\right)_S\right]_{S=S_i} = -\left[\left(\frac{\partial\Phi}{\partial Q}\right)_S\right]_{S=S_i}\left[\left(\frac{\partial Q}{\partial T}\right)_S\right]_{S=S_i}. \tag{2.29}$$

Using Eqs. (2.12) and (2.14) we further obtain

$$-\left[\left(\frac{\partial\Phi}{\partial T}\right)_S\right]_{S=S_i} = \frac{S_i}{Q_i}\left[1+\left(1+\frac{Q_i^2\pi^2}{b^2S_i^2}\right)\mathscr{F}\left(\frac{3}{4},1,\frac{5}{4},\frac{-Q_i^2\pi^2}{b^2S_i^2}\right)\right] \tag{2.30}$$

In order to calculate the rhs of the first Ehrenfest's Eq. (2.27), we adopt the following procedure. From Eqs. (2.17) and (2.20) we find

$$Q_i\beta = \left[\left(\frac{\partial Q}{\partial T}\right)_\Phi\right]_{S=S_i} = \left[\left(\frac{\partial Q}{\partial S}\right)_\Phi\right]_{S=S_i}\left(\frac{C_\Phi}{T_i}\right). \tag{2.31}$$

Therefore, the rhs of Eq. (2.27) becomes

$$\frac{\triangle C_\Phi}{T_iQ_i\triangle\beta} = \left[\left(\frac{\partial S}{\partial Q}\right)_\Phi\right]_{S=S_i}. \tag{2.32}$$

Using Eq. (2.14) we can further write

$$\frac{\triangle C_\Phi}{T_iQ_i\triangle\beta} = \frac{S_i}{Q_i}\left[1+\left(1+\frac{Q_i^2\pi^2}{b^2S_i^2}\right)\mathscr{F}\left(\frac{3}{4},1,\frac{5}{4},\frac{-Q_i^2\pi^2}{b^2S_i^2}\right)\right]. \tag{2.33}$$

From Eqs. (2.30) and (2.33) it is evident that both the lhs and the rhs of the first Ehrenfest's equation (Eq. (2.27)) indeed match at the critical points S_i. As a matter of fact, the divergence of C_Φ in the numerator is effectively canceled out by the diverging nature of β appearing at the denominator, which ultimately yields a finite value for the rhs of Eq. (2.27).

In order to calculate the lhs of the second Ehrenfest's Eq. (2.28), we use the thermodynamic relation

$$T = T(S, \Phi)$$

which leads to

$$\left(\frac{\partial T}{\partial \Phi}\right)_Q = \left(\frac{\partial T}{\partial S}\right)_\Phi \left(\frac{\partial S}{\partial \Phi}\right)_Q + \left(\frac{\partial T}{\partial \Phi}\right)_S . \tag{2.34}$$

Since C_Φ diverges at the critical points (S_i), it is evident from Eq. (2.17) that $\left[\left(\frac{\partial T}{\partial S}\right)_\Phi\right]_{S=S_i} = 0$. Also, from Eq. (2.14) we find that $\left(\frac{\partial S}{\partial \Phi}\right)_Q$ has finite values at the critical points (S_i). Thus from Eq. (2.34) and using Eq. (2.27) we may write

$$-\left[\left(\frac{\partial \Phi}{\partial T}\right)_Q\right]_{S=S_i} = -\left[\left(\frac{\partial \Phi}{\partial T}\right)_S\right]_{S=S_i} = \frac{\triangle C_\Phi}{T_i Q_i \triangle\beta}. \tag{2.35}$$

From Eq. (2.21), at the critical points we can write

$$\kappa Q_i = \left[\left(\frac{\partial Q}{\partial \Phi}\right)_T\right]_{S=S_i} . \tag{2.36}$$

Using Eqs. (2.23) and (2.20) this can be further written as

$$\kappa Q_i = -\left[\left(\frac{\partial T}{\partial \Phi}\right)_Q\right]_{S=S_i} Q_i \beta. \tag{2.37}$$

Therefore, the rhs of Eq. (2.28) may be finally expressed as[119]

$$\begin{aligned}\frac{\triangle\beta}{\triangle\kappa} &= -\left[\left(\frac{\partial \Phi}{\partial T}\right)_Q\right]_{S=S_i} \equiv -\left[\left(\frac{\partial \Phi}{\partial T}\right)_S\right]_{S=S_i} \\ &= \frac{S_i}{Q_i}\left[1 + \left(1 + \frac{Q_i^2\pi^2}{b^2 S_i^2}\right)\mathscr{F}\left(\frac{3}{4}, 1, \frac{5}{4}, \frac{-Q_i^2\pi^2}{b^2 S_i^2}\right)\right]. \end{aligned} \tag{2.38}$$

This as well proves the validity of the second Ehrenfest's equation (Eq. (2.28)) at the critical points S_i. Finally, using Eqs. (2.33) and (2.38), the Prigogine-Defay (PD) ratio[125] may be obtained as

$$\Pi = \frac{\triangle C_\Phi \triangle\kappa}{T_i Q_i (\triangle\beta)^2} = 1 \tag{2.39}$$

which confirms the second order nature of the phase transition.

2.5 Study of phase transition using state space geometry

Spurred by the elegance of the Riemannian geometry an alternative approach, commonly known as thermodynamic state space geometry, to describe ordinary thermodynamic systems as well as black holes (which also behave as thermodynamic systems) was put forward by G. Ruppeiner[129]-[135]. Encouraged by the successes of this approach in describing thermodynamic phases of several black hole systems[137]-[149], in this section we intend to apply this approach to study the phase transition phenomena in BI-AdS black hole. This will enable us to check the consistency and validity of the Ehrenfest's scheme that has been discussed in Section 2.4. The main theme of this approach has already been mentioned in some detailed in the Chapter 1. Based on that discussion we may infer that this method provides an elegant way to analyze a second order phase transition near the critical point.

In the state space geometry approach one aims to calculate the scalar curvature ($\mathscr{R}$)[129]-[135] which suffers a discontinuity at the critical point for the second order phase transition. In order to calculate the curvature scalar ($\mathscr{R}$) we need to determine the Ruppeiner metric coefficients which may be defined as[129, 131, 133, 135]

$$g_{ij}^{R} = -\frac{\partial^2 S(x^i)}{\partial x^i \partial x^j} \tag{2.40}$$

where $x^i = x^i(M,Q)$, $i = 1,2$ are the extensive variables of the system. From the computational point of view it is convenient to calculate the Weinhold metric coefficients[148]

$$g_{ij}^{W} = \frac{\partial^2 M(x^i)}{\partial x^i \partial x^j} \tag{2.41}$$

(where $x^i = x^i(S,Q)$, $i = 1,2$) that are conformally connected to that of the Ruppeiner geometry through the following map[136, 137]

$$dS_R^2 = \frac{dS_W^2}{T}. \tag{2.42}$$

Using these definitions (Eqs. (2.40),(2.41) and (2.42)) and choosing $x^1 = S$, $x^2 = Q$ the Ruppeiner metric coefficients are obtained as

$$g_{SS}^{R} = \frac{\frac{1}{2S}\left[-1 + \frac{3S}{\pi} + \frac{2b^2 S}{\pi}\left(1 - \sqrt{1 + \frac{Q^2\pi^2}{b^2 S^2}}\right) + \frac{4\pi Q^2}{S} - \frac{2\pi^3 Q^4}{b^2 S^3}\right]}{\left[1 + \frac{3S}{\pi} + \frac{2b^2 S}{\pi}\left(1 - \sqrt{1 + \frac{Q^2\pi^2}{b^2 S^2}}\right)\right]}, \tag{2.43}$$

$$g_{SQ}^{R} = \frac{\left(\frac{-2\pi Q}{S} + \frac{\pi^3 Q^3}{b^2 S^3}\right)}{\left[1 + \frac{3S}{\pi} + \frac{2b^2 S}{\pi}\left(1 - \sqrt{1 + \frac{Q^2\pi^2}{b^2 S^2}}\right)\right]}, \tag{2.44}$$

$$g_{QQ}^{R} = \frac{\left(4\pi - \frac{6\pi^3 Q^2}{5b^2 S^2}\right)}{\left[1 + \frac{3S}{\pi} + \frac{2b^2 S}{\pi}\left(1 - \sqrt{1 + \frac{Q^2\pi^2}{b^2 S^2}}\right)\right]}. \tag{2.45}$$

In obtaining these metric coefficients we have used Eq. (2.9). Now, the Ricci curvature scalar ($\mathscr{R}$) can be computed as[129, 131, 133, 135]

$$\mathscr{R} = \frac{\wp(S,Q)}{\Re(S,Q)}. \tag{2.46}$$

The numerator $\wp(S,Q)$ is too much cumbersome which prevents us to present its detail expression for the present work. However, the denominator $\Re(S,Q)$ may be expressed as

$$\Re(S,Q) = \sqrt{1+\frac{\pi^2Q^2}{b^2S^2}}\left[-\pi+\left(-3+2b^2\left(-1+\sqrt{1+\frac{\pi^2Q^2}{b^2S^2}}\right)\right)S\right]\left(\pi^2Q^2+b^2S^2\right)$$
$$\Bigg(-\pi^6Q^6+12b^2\pi^4Q^4S^2-3b^2\pi^3Q^2S^3+b^2\pi^2Q^2\left(9-2b^2\left(7+3\sqrt{1+\frac{\pi^2Q^2}{b^2S^2}}\right)\right)$$
$$S^4+10b^4\pi S^5+10b^4\left(-3+2b^2\left(-1+\sqrt{1+\frac{\pi^2Q^2}{b^2S^2}}\right)\right)S^6\Bigg)^3. \tag{2.47}$$

In Fig. 2.8 and Fig. 2.9 below we have plotted the thermodynamic scalar curvature ($\mathscr{R}$) as a function of entropy (S) as an important part of our discussion.

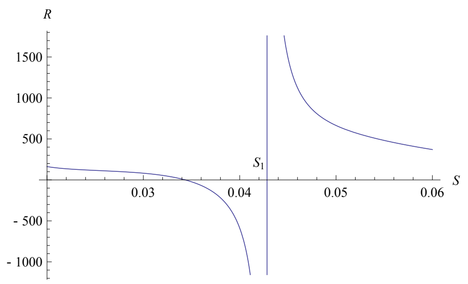

Figure 2.8: *Plot of scalar curvature ($\mathscr{R}$) against entropy (S) at the first point of divergence (S_1) (antisymmetric divergence), for fixed $Q = 0.13$ and $b = 10$.*

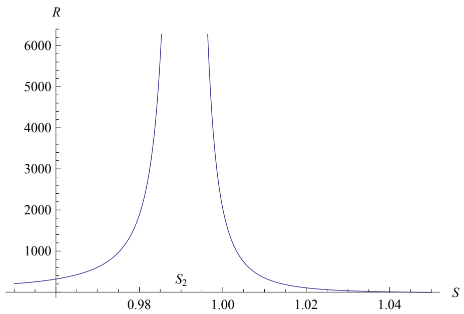

Figure 2.9: *Plot of scalar curvature ($\mathscr{R}$) against entropy (S) at the second point of divergence (S_2) (symmetric divergence), for fixed $Q = 0.13$ and $b = 10$.*

From Figs. 2.8 and 2.9 we observe that the scalar curvature ($\mathscr{R}$) diverges exactly at the points where the specific heat (C_Φ) diverges (see Figs. 2.2 & 2.3). This is an expected result, since according to Ruppeiner's prescription any divergence in $\mathscr{R}$ implies a corresponding divergence in the heat capacity C_Φ (which is thermodynamic analog of C_P) that essentially leads to a change in stability[133]. On the other hand, no divergence in $\mathscr{R}$ could be observed at the Davis critical point[70], which is marked by the divergence in C_Q (which is the thermodynamic analog of C_V). Similar features have also been observed earlier[116, 117].

2.6 Conclusive remarks

In this chapter we systematically analyze the phase transition phenomena in Born-Infeld AdS (BI-AdS) black holes. We have performed our entire analysis based on standard thermodynamic approach where the standard thermodynamic variables have been suitably modified so that they can be applied to our black hole system. In this regard the laws of black hole mechanics have been our guiding principle[59]. The main facet of our model is the presence of a non-linear higher derivative correction to the Maxwell gauge field (which is coupled to the gravity action, Eq. (2.4)) known as the Born-Infeld correction. Notably, this correction results non-trivial modifications of the geometric and thermodynamic properties of the black hole (see Section 2.3.1 & Section 2.3.2). Our results are valid for all orders in the Born-Infeld (BI) parameter (b). The continuous nature of the $T - S$ plot (Fig. 2.1) essentially rules out the possibility of any first order transition. On the other hand, the discontinuity of the heat capacity C_Φ (Fig. 2.2 & Fig. 2.3) indicates the onset of a continuous higher order transition. In order to address this issue further, we provide a detailed analysis of the phase transition phenomena using Ehrenfest's scheme of standard thermodynamics[60] which uniquely determines the second order nature of the phase transition. At this stage it is reassuring to note that the first application of the Ehrenfest's scheme in order to determine the nature of phase transition in charged (Reissener-Nordstrom (RN-AdS)) AdS black holes had been commenced in Ref.[117]. There the analysis had been carried out *numerically* in order to check the validity of the Ehrenfest's equations close to the critical point(s). Unfortunately, the analysis presented in Ref.[117] was actually in an underdeveloped stage. This is mainly due to the fact that at that time no such analytic scheme was available in the literature. As a result, at that stage of analysis it was not possible to check the validity of the Ehrenfest's equations exactly at the critical point(s) due to the occurrence infinite divergences of various thermodynamic entities at the (phase) transition point(s).

In order to remove the above mentioned difficulty and put the Ehrenfest's scheme in a firm theoretical platform, we have provided an analytic scheme to check the validity of the Ehrenfest's equations exactly at the critical point(s). Furthermore, we have carried out the entire analysis taking the particular example of BI-AdS black hole which is basically the non-linear generalization of RN-AdS black hole. Therefore, our results are quite general, and hence are valid for a wider class of charged black holes in the usual Einstein gravity.

Also, we have analyzed the phase transition phenomena using state space geometry approach. Our analysis shows that the scalar curvature ($\mathscr{R}$) suffers discontinuities exactly at the (critical) points where the heat capacity (C_Φ) diverges (Fig. 2.8 & Fig. 2.9). This further indicates the second order nature of the phase transition. Thus, from our analysis it is clear that both the Ehrenfest's scheme and the state space geometry approach essentially lead to an identical conclusion. This also establishes their compatibility while studying phase transitions in black holes.

Finally, we remark that the curvature scalar, which behaves in a very suggestive way for conventional systems, displays similar properties for black holes. Specifically, a diverging curvature that signals the occurrence of a second order phase transition in usual systems retains this characteristic for black holes. Our analysis thus reveals a direct connection between the Ehrenfest's scheme and the well established thermodynamic state space geometry.

In this chapter we have successfully addressed the nature of phase transition in charged black holes with higher derivative corrections to the coupled gauge field. But, there remains several other crucial aspects that one encounters while carrying out the study of thermodynamics of black holes. The study of critical phenomena in black holes is one of them. Studying this phenomena one becomes familiar with the behavior of various thermodynamic systems close to the critical point(s). Keeping in mind the fact that black holes are thermodynamic systems, in the next Chapter 3 we shall study critical phenomena in a higher curvature charged AdS black hole, namely, the third-order Lovelock-AdS black hole. In addition to the curvature correction to the Einstein gravity the Born-Infeld correction to the usual Maxwell gauge filed will also be taken into account. In other words, we'll study the thermodynamic aspect of a black hole with both non-linear gauge and gravity corrections.

Chapter 3

Critical Phenomena In Higher Curvature Charged AdS Black Holes

3.1 Overview

In the study of phase transition of a given thermodynamic system one becomes familiar with the behavior of the system in the neighborhood of the critical point(s). These are the points of singularity of various thermodynamic quantities (such as, specific heat, compressibility, etc.) related to the system. The diverging thermodynamic quantities are derivatives of the free energy associated with the system, and the order of the derivatives determine the order of the phase transition.[1] In fact, the divergence of the correlation length (ξ) of the system is manifested as the divergences of these quantities[150]-[152]. The primary motivation for studying critical phenomena of a system is to express these singularities in the form of power laws, characterized by a set of indices, known as the static critical exponents, that determine the critical behavior of the given system qualitatively[150]. It is found that, as $\xi \to \infty$, correlations extend over the macroscopic distances in the system. As a result, two different systems with different microscopic structures can no longer be differentiated from each other leading to a universal behavior of the systems[150]. In general, the critical exponents depend on *(i)* the spatial dimensionality (n) of the space in which the system is embedded, *(ii)* the range of interactions in the system, etc. For example, systems which posses short range interactions the critical exponents are found to be dependent on n, whereas for systems having long range interactions these are independent of n. It is interesting to note that these exponents are not independent of each other, they are related to each other by the *static scaling laws* [151, 152]. Since black holes behave as thermodynamic systems, it is quite natural to study the qualitative behavior of the black holes via the critical phenomena and determine the critical exponents associated with the thermodynamic quantities

[1] A brief overview of the phase transition phenomena has been given in Section 2.2, Chapter 2.

pertaining to the black holes[153]-[169].

Following this line of argument, in this chapter we intend to carry out a detailed analytic computation to investigate the critical phenomena in charged third order Lovelock black hole in AdS space-time. Apart from the higher curvature corrections, we introduce Born-Infeld corrections to the matter sector of the mentioned black hole. Interestingly, since we are considering third order Lovelock black holes, we are in fact in the regime of higher dimensional gravity theory. A brief discussion on the importance of gravity theories in higher dimensions is given in Chapter 1. However, some additional motivations for the present study are in order: (i) The violation of the usual *area law* of black hole entropy is a general feature of higher curvature black holes. In this regard it is interesting to study the thermodynamic phase structure of the third order Lovelock-Born-Infeld-AdS (henceforth LBI-AdS) black hole and verify whether it possesses the same set of critical exponents as was obtained earlier in Ref.[167] in the context of higher dimensional charged AdS black holes; (ii) Born-Infeld-AdS black holes in the usual $(3 + 1)$-dimensional Einstein gravity exhibit a lower bound on the Born-Infeld parameter[110]. However, in Ref.[167] it was observed that this bound was removed in higher dimensions. Thus, it is worthy of checking whether there is any bound on the Born-Infeld parameter in the third order LBI-AdS black hole we are considering.

Enlightened by these facts, we primarily address the following issues in the context of LBI-AdS black hole which may be viewed as a solution to a higher dimensional gravity theory where non-linear, higher derivative corrections to both gauge and gravity sector is explicit. These are given by: (i) Phase structure and stability of the LBI-AdS black hole, (ii) the critical exponents associated with relevant thermodynamic quantities of the black hole, and (iii) validity of the static scaling laws and the static scaling hypothesis for this model.

The contents of the present chapter are organized as follows: In Section 3.2 we discuss geometric and thermodynamic properties of the third order LBI-AdS black hole. The phase structure and stability of the mentioned black hole are analyzed qualitatively in Section 3.3. In Section 3.4 the critical exponents are computed, and the validity of the static scaling laws and static scaling hypothesis are discussed. Finally, we end the chapter with some conclusive remarks in Section 3.5.

3.2 Geometric and thermodynamic properties of Lovelock-Born-Infeld-AdS black holes

3.2.1 Gravity action and metric structure

Lovelock black holes are the solutions to the Lovelock gravity which is essentially Einstein gravity modified by higher curvature terms coupled to the usual Einstein-Hilbert action[5]-[8]. The effective action that describes the Lovelock gravity in

$d=(n+1)$-dimensions (n: spatial dimensions) can be written as[2][5]

$$\mathscr{I}=\frac{1}{16\pi}\int d^{n+1}x\sqrt{-g}\sum_{i=0}^{[\frac{n+1}{2}]}\alpha_i\mathscr{L}_i \tag{3.1}$$

where α_i are arbitrary constants and $\mathscr{L}_i$ is the *Euler density* of a $2i$-dimensional manifold (see the discussions of Chapter 1). In $(n+1)$ dimensions all terms for which $i>[(n+1)/2]$ are equal to zero, the term $i=(n+1)/2$ is a topological term, and terms for which $i<[(n+1)/2]$ contribute to the field equations. Since we are studying third order Lovelock gravity in the presence of Born-Infeld nonlinear electrodynamics[20], the effective action of Eq. (3.1) may be written as[102]

$$\begin{aligned}\mathscr{I}&=\frac{1}{16\pi}\int d^{n+1}x\sqrt{-g}\Big(\alpha_0\mathscr{L}_0+\alpha_1\mathscr{L}_1+\alpha_2\mathscr{L}_2+\alpha_3\mathscr{L}_3+L(F)\Big)\\&=\frac{1}{16\pi}\int d^{n+1}x\sqrt{-g}\Big(-2\Lambda+\mathscr{R}+\alpha_2\mathscr{L}_2+\alpha_3\mathscr{L}_3+L(F)\Big).\end{aligned} \tag{3.2}$$

In Eq. (3.2) Λ is the cosmological constant given by $\Lambda=-n(n-1)/2L^2$ (L: the AdS length), α_2 and α_3 are the second and third order Lovelock coefficients, $\mathscr{L}_1=\mathscr{R}$ is the usual Einstein-Hilbert Lagrangian, $\mathscr{L}_2=\big(R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta}-4R_{\mu\nu}R^{\mu\nu}+\mathscr{R}^2\big)$ is the Gauss-Bonnet Lagrangian, and

$$\begin{aligned}\mathscr{L}_3&=2R^{\mu\nu\sigma\kappa}R_{\sigma\kappa\rho\tau}R^{\rho\tau}{}_{\mu\nu}+8R^{\mu\nu}{}_{\sigma\rho}R^{\sigma\kappa}{}_{\nu\tau}R^{\rho\tau}{}_{\mu\kappa}+24R^{\mu\nu\sigma\kappa}R_{\sigma\kappa\nu\rho}R^{\rho}{}_{\mu}\\&+3\mathscr{R}R^{\mu\nu\sigma\kappa}R_{\sigma\kappa\mu\nu}+24R^{\mu\nu\sigma\kappa}R_{\sigma\mu}R_{\kappa\nu}+16R^{\mu\nu}R_{\nu\sigma}R^{\sigma}{}_{\mu}-12\mathscr{R}R^{\mu\nu}R_{\mu\nu}+\mathscr{R}^3\end{aligned} \tag{3.3}$$

is the third order Lovelock Lagrangian.

The non-linear Born-Infeld Lagrangian $L(F)$ in Eq. (3.2) is given by[102]

$$L(F)=4b^2\left(1-\sqrt{1+\frac{F^2}{2b^2}}\right). \tag{3.4}$$

Note that, in Eq. (3.4), $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$, $F^2=F_{\mu\nu}F^{\mu\nu}$, and b is the Born-Infeld parameter. In the limit $b\to\infty$ we recover the standard Maxwell form $L(F)=-F^2$. It is to be remembered that, in all our subsequent analysis we shall consider the special case $\alpha_3=2\alpha_2^2=\frac{\alpha'^2}{72}$ [6]-[8],[101, 102].

The solution of the LBI-AdS black hole in $d=(n+1)$-dimensions can be written as[102]

$$ds^2=-f(r)dt^2+\frac{1}{f(r)}dr^2+r^2d\Omega^2_{k,n-1} \tag{3.5}$$

[2]Here we are using the system of natural units in which $G=\hbar=k_B=c=1$.

where

$$d\Omega^2_{k,n-1} = \begin{cases} d\theta_1^2 + \sum_{i=2}^{n-1}\prod_{j=1}^{i-1} sin^2\theta_j d\theta_i^2, & k=1 \\ d\theta_1^2 + sinh^2\theta_1 d\theta_2^2 + sinh^2\theta_1 \sum_{i=3}^{n-1}\prod_{j=2}^{i-1} sin^2\theta_j d\theta_i^2, & k=-1 \\ \sum_{i=1}^{n-1} d\phi_i^2, & k=0. \end{cases} \tag{3.6}$$

In Eq. (3.6) the quantity k determines the structure of the black hole horizon: $k = +1$, -1 and 0 correspond to spherical, hyperbolic and planar horizon, respectively. At this point of discussion it must be mentioned that the Lagrangian in Eq. (3.2) is the most general Lagrangian in seven space-time dimensions that produces the second order field equations[101]. Thus, we shall restrict ourselves in the seven space-time dimensions ($n = 6$).

The equation of motion for the electromagnetic field for this $(6+1)$-dimensional space-time can be obtained by varying the action Eq. (3.2) with respect to the gauge field A_μ. The result is[102]:

$$\partial_\mu \left(\frac{\sqrt{-g}F^{\mu\nu}}{\sqrt{1+\frac{F^2}{2b^2}}} \right) = 0. \tag{3.7}$$

The solution of Eq. (3.7) is given by[102]

$$A_\mu = -\sqrt{\frac{5}{8}} \left(\frac{q}{r^4}\right) \mathscr{H}(\eta)\delta^0_\mu. \tag{3.8}$$

Here $\mathscr{H}(\eta)$ is the abbreviation of the hypergeometric function given by[291]

$$\mathscr{H}(\eta) = \mathscr{H}\left(\frac{1}{2}, \frac{2}{5}, \frac{7}{5}, -\eta\right). \tag{3.9}$$

In Eq. (3.8), $\eta = \dfrac{10q^2}{b^2r^{10}}$ and q is a constant of integration which is related to the charge (Q) of the black hole. We can obtain the charge (Q) of the black hole by calculating the flux of the electric field at infinity[1, 101, 102]. Therefore

$$\begin{aligned} Q &= -\int_{\mathscr{B}} d^{n-1}\omega\sqrt{\sigma}n_\mu\tau_\nu \left(\frac{F^{\mu\nu}}{\sqrt{1+\frac{F^2}{2b^2}}}\right) \\ &= \frac{\mathscr{V}_{n-1}}{4\pi}\sqrt{\frac{(n-1)(n-2)}{2}}q \\ &= \frac{\sqrt{10}\pi^2 q}{4} \qquad (\text{for } \ n=6) \end{aligned} \tag{3.10}$$

where n_μ and τ_ν are the time-like and space-like unit normal vectors to the boundary $\mathscr{B}$, respectively, and σ is the determinant of the induced metric σ_{ij} on $\mathscr{B}$ having coordinates ω^i. It is to be noted that, in deriving Eq. (3.10) we have only considered the F^{tr} component of $F^{\mu\nu}$.

The quantity $\mathscr{V}_{n-1}$ in Eq. (3.10) is the volume of the $(n-1)$ sphere and may be written as

$$\mathscr{V}_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{3.11}$$

The metric function $f(r)$ in Eq. (3.5) can be rewritten as[102]

$$f(r) = k + \frac{r^2}{\alpha'}\left(1 - \chi(r)^{\frac{1}{3}}\right) \tag{3.12}$$

where

$$\chi(r) = 1 + \frac{3\alpha' m}{r^6} - \frac{2\alpha' b^2}{5}\left[1 - \sqrt{1+\eta} - \frac{\Lambda}{2b^2} + \frac{5\eta}{4}\mathscr{H}(\eta)\right]. \tag{3.13}$$

3.2.2 Thermodynamic quantities

In the previous Chapter 2 the importance of various thermodynamic quantities in the study of thermodynamic properties of black holes were established. In this chapter we also intend to discuss various thermodynamic aspects of a black hole with non-linear corrections. Thus, it is necessary to compute important thermodynamic quantities for the LBI-AdS black hole whose behavior (at the critical point(s) of phase transition) will be studied in this chapter.

To begin with, the *quasilocal energy* M of asymptotically AdS black holes may be obtained by using the *counterterm* method which is indeed inspired by the AdS/CFT correspondence[170, 171]. This method was applied in Lovelock gravity to compute the associated conserved quantities[101, 102, 103, 104, 182]. However, for the asymptotically AdS solutions of the third order Lovelock black holes the action may be written as[102, 103, 182]

$$\mathscr{A} = \mathscr{I} + \underbrace{\frac{1}{8\pi}\int_{\partial\mathscr{M}} d^n x\sqrt{|\gamma|}\left(\mathscr{I}_b^1 + \alpha_2\mathscr{I}_b^2 + \alpha_3\mathscr{I}_b^3\right)}_{\text{boundary terms}} + \underbrace{\frac{1}{8\pi}\int_{\partial\mathscr{M}} d^n x\sqrt{|\gamma|}\left(\frac{n-1}{L}\right)}_{\text{counterterm}}, \tag{3.14}$$

where γ is the determinant of the induced metric γ_{ab} on the time-like boundary $\partial\mathscr{M}$ of the space-time manifold $\mathscr{M}$. The quantity L is a new scale length factor given by

$$L = \frac{15\sqrt{\alpha'(1-\lambda)}}{5 + 9\alpha' - \lambda^2 - 4\lambda}, \tag{3.15}$$

where

$$\lambda = \left(1 - 3\alpha'\right)^{\frac{1}{3}}. \tag{3.16}$$

The boundary terms in Eq. (3.14) are chosen such that the action possesses well defined variational principle, whereas, the counterterm makes the action and the associated conserved quantities finite. These terms can be identified as[102, 103, 182]

$$\mathscr{I}_b^1 = K \tag{3.17}$$

$$\mathscr{I}_b^2 = 2\Big(J - 2\bar{G}^1_{ab}K^{ab}\Big) \tag{3.18}$$

$$\begin{aligned}\mathscr{I}_b^3 = 3\Big(&P - 2\bar{G}^2_{ab}K^{ab} - 12\bar{R}_{ab}J^{ab} + 2\bar{\mathscr{R}}J \\ &- 4K\bar{R}_{abcd}K^{ac}K^{bd} - 8K\bar{R}_{abcd}K^{ac}K^b_eK^{ed}\Big).\end{aligned} \tag{3.19}$$

Here K is the trace of the extrinsic curvature K_{ab}. In Eq. (3.18) $\bar{G}^1_{ab}$ is the Einstein tensor for γ_{ab} in n-dimensions and J is the trace of the following tensor:

$$J_{ab} = \frac{1}{3}\Big(2KK_{ac}K^c_b + K_{cd}K^{cd}K_{ab} - 2K_{ac}K^{cd}K_{db} - K^2K_{ab}\Big). \tag{3.20}$$

In Eq. (3.19) $\bar{G}^2_{ab}$ is the second order Lovelock tensor for γ_{ab} in n-dimensions which is given by

$$\bar{G}^2_{ab} = 2\Big(\bar{R}_{acde}\bar{R}_b{}^{cde} - 2\bar{R}_{afbc}\bar{R}^{fc} - 2\bar{R}_{ac}\bar{R}^c{}_b + \bar{\mathscr{R}}\bar{R}_{ab}\Big) - \mathscr{L}_2(\gamma)\gamma_{ab}, \tag{3.21}$$

whereas, P is the trace of

$$\begin{aligned}P_{ab} = \frac{1}{5}\Bigg(&\Big[K^4 - 6K^2K^{cd}K_{cd} + 8KK_{cd}K^d_eK^{ec} - 6K_{cd}K^{de}K_{ef}K^{fc} + 3\Big(K_{cd}K^{cd}\Big)^2\Big]K_{ab} \\ &- \Big(4K^3 - 12KK_{ed}K^{ed} + 8K_{de}K^e_fK^{fd}\Big)K_{ac}K^c_b - 24KK_{ac}K^{cd}K_{de}K^e_b \\ &+ \Big(12K^2 - 12K_{ef}K^{ef}\Big)K_{ac}K^{cd}K_{db} + 24K_{ac}K_{cd}K_{de}K^{ef}K_{fb}\Bigg).\end{aligned} \tag{3.22}$$

Next, using the method prescribed in Ref.[75], we obtain the divergence-free energy-momentum tensor as[102, 103, 182]

$$T^{ab} = \frac{1}{8\pi}\Big[\Big(K^{ab} - K\gamma^{ab}\Big) + 2\alpha_2\Big(3J^{ab} - J\gamma^{ab}\Big) + 3\alpha_3\Big(5P^{ab} - P\gamma^{ab}\Big) + \frac{n-1}{L}\gamma^{ab}\Big]. \tag{3.23}$$

The first three terms of Eq. (3.23) result from the variation of the boundary terms of Eq. (3.14) with respect to the induced metric γ^{ab}, whereas, the last term is obtained by considering the variation of the counterterm of Eq. (3.14) with respect to γ^{ab}.

For any space-like surface $\mathscr{B}$ in $\partial\mathscr{M}$ which has the metric σ_{ij} we can write the boundary metric in the following form[75, 102, 103, 182]:

$$\gamma_{ab}dx^a dx^b = -N^2dt^2 + \sigma_{ij}\left(d\omega^i + V^i dt\right)\left(d\omega^j + V^j dt\right), \tag{3.24}$$

where ω^i are coordinates on $\mathscr{B}$, N and V^i are the lapse function and the shift vector, respectively. For any Killing vector field ξ on the the space-like boundary $\mathscr{B}$ in $\partial\mathscr{M}$, we may write the conserved quantities associated with the energy momentum tensor (T^{ab}) as[75, 102, 103, 182]

$$\mathscr{Q}_\xi = \int_{\mathscr{B}} d^{n-1}\omega\sqrt{\sigma}\; n^a\xi^b T_{ab} \tag{3.25}$$

where σ is the determinant of the metric σ_{ij} on $\mathscr{B}$, n^a is the time-like unit normal to $\mathscr{B}$, and $\xi^b \left(=\frac{\partial}{\partial t}\right)$ is the time-like Killing vector field. For the metric Eq. (3.5) we can write $n^a = (1/\sqrt{f(r)}, 0, 0, 0, ...)$, $\xi^a = (1, 0, 0, ...)$, and $K_{ab} = -\gamma_a^m \nabla_m \tau_b$, where $\tau^a = (0, f(r), 0, ...)$ is the space-like unit normal to the boundary.

With these values of n^a and ξ^a the only nonvanishing component of T_{ab} becomes T_{00}. Hence, $\mathscr{Q}_\xi$ corresponds to the *quasilocal energy* M of the black hole. Thus, from Eq. (3.25) we obtain the expression for the *quasilocal energy* of the black hole as

$$M = \int_{\mathscr{B}} d^{n-1}\omega\sqrt{\sigma}\; n^0\xi^0 T_{00}. \tag{3.26}$$

Using Eqs. (3.12) and (3.23), Eq. (3.26) can be computed as

$$\begin{aligned} M\Big|_{n=6} &= \frac{\mathscr{V}_{n-1}}{16\pi}(n-1)\, m\Big|_{n=6} \\ &= \frac{5\pi^2}{16} m, \end{aligned} \tag{3.27}$$

where the constant m is expressed as the real root of the equation

$$f(r = r_+) = 0. \tag{3.28}$$

Using Eq. (3.11) and substituting m from Eq. (3.28) we finally obtain from Eq. (3.27)

$$M = \frac{5\pi^2}{16}\left[\frac{k^3\alpha'^2}{3} + kr_+^4 + k^2\alpha' r_+^2 + \frac{2b^2 r_+^6}{15}\left(1 - \sqrt{1+\eta_+} - \frac{\Lambda}{2b^2} + \frac{20Q^2}{b^2\pi^4 r_+^{10}}\mathscr{H}(\eta_+)\right)\right]. \tag{3.29}$$

The electrostatic potential difference between the black hole horizon and the infinity may be defined as[102]

$$\begin{aligned} \Phi &= \sqrt{\frac{(n-1)}{2(n-2)}}\frac{q}{r_+^{n-2}}\mathscr{H}(\eta_+) \\ &= \frac{Q}{\pi^2 r_+^4}\mathscr{H}(\eta_+) \qquad (\text{for } n = 6) \end{aligned} \tag{3.30}$$

where

$$\eta_+ = \frac{16Q^2}{b^2\pi^4 r_+^{10}}. \tag{3.31}$$

It is to be noted that in obtaining Eq. (3.31) we have used Eq. (3.10).

Following the method discussed in Chapter 2 (see Section 2.3.2, Eq. (2.10)), the Hawking temperature of the third order LBI-AdS black hole can be obtained as

$$T = \frac{10kr_+^4 + 5k\alpha' r_+^2 + 2b^2 r_+^6 \left(1 - \sqrt{1 + \frac{16Q^2}{b^2\pi^4 r_+^{10}}}\right) - \Lambda r_+^6}{10\pi r_+ (r_+^2 + k\alpha')^2}. \tag{3.32}$$

In order to calculate the entropy of the third order LBI-AdS black hole we make use of the first law of black hole thermodynamics[59]:

$$dM = TdS + \Phi dQ. \tag{3.33}$$

In fact, it has been found that the thermodynamic quantities (e.g., entropy, temperature, quasilocal energy, etc.) of the LBI-AdS black holes satisfy the first law of black hole mechanics[94, 101, 102, 103, 104].

Using Eq. (3.33) we obtain the entropy of the black hole as,

$$\begin{aligned} S &= \int_0^{r_+} \frac{1}{T}\left(\frac{\partial M}{\partial r_+}\right)_Q dr_+ \\ &= \frac{\pi^3}{4}\left(r_+^5 + \frac{10k\alpha' r_+^3}{3} + 5k^2\alpha'^2 r_+\right) \end{aligned} \tag{3.34}$$

where we have used Eqs.(3.29) and (3.32). Interestingly, identical expression for the entropy can be obtained by using somewhat different approach[61]-[65]. In this approach, in an arbitrary spatial dimension n, the expression for the *Wald entropy* for higher curvature black holes is written as

$$\begin{aligned} S &= \frac{1}{4}\sum_{j=1}^{[\frac{n-2}{2}]} j\alpha_j \int_{\mathscr{B}} d^{n-1}\omega\sqrt{|\sigma|}\,\mathscr{L}_{j-1}(\sigma) \\ &= \frac{1}{4}\int_{\mathscr{B}} d^{n-1}\omega\sqrt{|\sigma|}\Big(1 + 2\alpha_2\tilde{\mathscr{R}} + 3\alpha_3(\tilde{R}_{abcd}\tilde{R}^{abcd} - 4\tilde{R}_{ab}\tilde{R}^{ab} + \tilde{\mathscr{R}}^2)\Big) \\ &= \frac{\mathscr{V}_{n-1}}{4}\left[r_+^4 + \frac{2(n-1)}{(n-3)}k\alpha' r_+^2 + \frac{(n-1)}{(n-5)}k^2\alpha'^2\right] r_+^{n-5}, \end{aligned} \tag{3.35}$$

where $\mathscr{L}_j(\sigma)$ is the jth order Lovelock Lagrangian of σ_{ij} and the *tilde* denotes the corresponding quantities for the induced metric σ_{ij}. If we put $n = 6$ in Eq. (3.35) we get back the expression of Eq. (3.34). At this stage it is worth mentioning that the entropy of black holes both in the usual Einstein gravity and in higher curvature gravity can be obtained by using the approach of Refs.[66, 67] and [68], respectively. The expression for the entropy of the third order Lovelock black hole as given by Eq. (3.34) is the same as that of Ref.[68]. Thus, we may infer that the entropy of the black hole obtained from the first law of black hole mechanics is indeed the Wald entropy.

Let us discuss the limit $\alpha' \to 0$. In this limit the corresponding expression for the Hawking temperature and entropy of a (6 + 1)-dimensional Born-Infeld AdS (BI-AdS) black hole can be recovered from Eq. (3.32) and Eq. (3.34) as[167]

$$T_{BI-AdS} = \frac{1}{4\pi}\left[\frac{4k}{r_+} + \frac{4b^2 r_+}{5}\left(1 - \sqrt{1 + \frac{Q^2}{b^2 r_+^4}}\right) - \frac{2\Lambda r_+}{5}\right], \tag{3.36}$$

$$S = \frac{\pi^3}{4} r_+^5, \tag{3.37}$$

respectively.

From Eq. (3.34) (or Eq. (3.35)) we find that the entropy is not proportional to the one-fourth of the horizon area as in the case of the black holes in the Einstein gravity. In fact, this is one of the main features of the higher curvature black holes − the violation of the usual *area law* of black holes in Einstein gravity[19].

In our study of critical phenomena we will be mainly concerned about the spherically symmetric space-time. In this regard we will always take the value of k to be +1. Substituting $k = 1$ in Eqs.(3.29), (3.32) and (3.34) we finally obtain the expressions for the quasilocal energy, the Hawking temperature and the entropy of the third order LBI-AdS black hole as

$$M = \frac{5\pi^2}{16}\left[\frac{\alpha'^2}{3} + r_+^4 + \alpha' r_+^2 + \frac{2b^2 r_+^6}{15}\left(1 - \sqrt{1+\eta_+} - \frac{\Lambda}{2b^2} + \frac{20Q^2}{b^2\pi^4 r_+^{10}}\mathscr{H}(\eta_+)\right)\right], \tag{3.38}$$

$$T = \frac{10r_+^4 + 5\alpha' r_+^2 + 2b^2 r_+^6\left(1 - \sqrt{1 + \frac{16Q^2}{b^2\pi^4 r_+^{10}}}\right) - \Lambda r_+^6}{10\pi r_+ (r_+^2 + \alpha')^2}, \tag{3.39}$$

$$S = \frac{\pi^3}{4}\left(r_+^5 + \frac{10\alpha' r_+^3}{3} + 5\alpha'^2 r_+\right). \tag{3.40}$$

3.3 Phase structure and stability of third order LBI-AdS black hole

In this section we aim to discuss the nature of phase transition and stability of the third order LBI-AdS black hole. A powerful method, based on the Ehrenfest's scheme of ordinary thermodynamics, was introduced by the authors of Ref.[115] in order to determine the nature of phase transition in black holes. Using this analytic method phase transition phenomena in various AdS black holes were explored[115]-[120], and this was discussed elaborately in the previous Chapter 2 based on Ref.[120]. Phase transition in higher dimensional AdS black holes has been discussed in Ref.[119] as well.

In this work we qualitatively discuss the phase transition phenomena in the LBI-AdS black hole following the arguments presented in the above mentioned works.

However, in this work, the analysis presented in Refs.[115]-[120] has been extended to include higher curvature terms for the first time taking the particular example of LBI-AdS black hole. Nevertheless, we will analyze the phase transition phenomena in this black hole only *qualitatively*, no quantitative discussion will be presented in this regard.

To analyze the nature of phase transition we plot the Hawking temperature (Eq. (3.39)) as a function of horizon radius r_+.

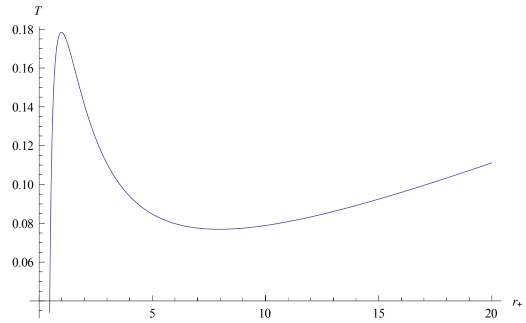

Figure 3.1: *Plot of Hawking temperature (T) against horizon radius (r_+), for* $\alpha' = 0.5$, $Q = 0.50$ *and* $b = 10$.

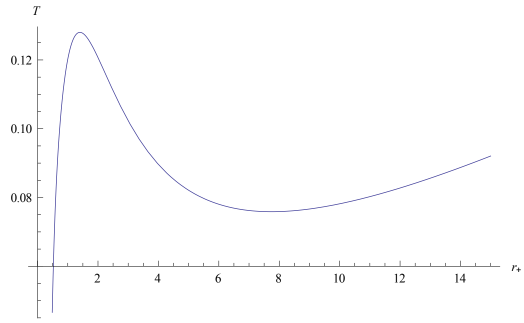

Figure 3.2: *Plot of Hawking temperature (T) against horizon radius (r_+), for* $\alpha' = 1.0$, $Q = 0.50$ *and* $b = 10$.

From these figures it is evident that there is no discontinuity in the temperature of the black hole. This rules out the possibility of first order phase transition[115]-[120] (see also Chapter 2, Section 2.3.2).

In order to see whether there is any higher order phase transition, we need to calculate the specific heat of the black hole. In the *canonical ensemble* framework the specific heat at constant charge, C_Q, (This is analogous to the specific heat heat at constant volume (C_V) in the ordinary thermodynamics.) can be calculated as[166, 167]

$$C_Q = T\left(\frac{\partial S}{\partial T}\right)_Q = T\frac{(\partial S/\partial r_+)_Q}{(\partial T/\partial r_+)_Q} = \frac{\mathscr{N}(r_+,Q)}{\mathscr{D}(r_+,Q)} \tag{3.41}$$

where

$$\mathscr{N}(r_+,Q) = \frac{5}{4}\pi^7 r_+^5\left(r_+^2+\alpha'\right)^3\sqrt{1+\frac{16Q^2}{b^2\pi^4 r_+^{10}}}\left[10r_+^2+5\alpha'+2b^2r_+^4\left(1-\sqrt{1+\frac{16Q^2}{b^2\pi^4 r_+^{10}}}\right)\right.$$
$$\left.-\Lambda r_+^4\right], \tag{3.42}$$

$$\mathscr{D}(r_+, Q) = 128Q^2 + \left[15\pi^4 r_+^6 \alpha' + 5\pi^4 r_+^4 \alpha'^2 - \Lambda\pi^4 r_+^{10} - 5\pi^4 r_+^8 (2 + \alpha'\Lambda)\right] \sqrt{1 + \frac{16Q^2}{b^2\pi^4 r_+^{10}}} - \left(2b^2\pi^4 r_+^{10} + 10b^2\pi^4 r_+^8 \alpha'\right)\left(1 - \sqrt{1 + \frac{16Q^2}{b^2\pi^4 r_+^{10}}}\right). \tag{3.43}$$

In the derivation of Eq. (3.41) we have used Eqs. (3.39) and (3.40).

We plot the specific heat (C_Q) as a function of the horizon radius (r_+) in Fig. 3.3-Fig. 3.10 below. Here we have zoomed in the plots near the two critical points r_i ($i = 1, 2$) separately for convenience.

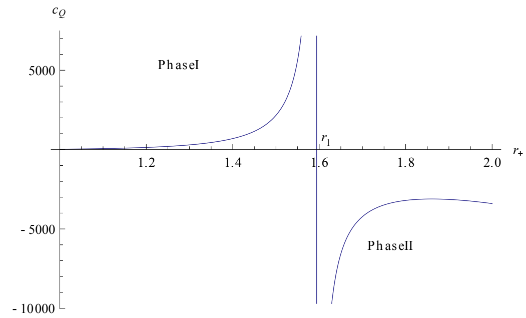

Figure 3.3: *Plot of specific heat (C_Q) against horizon radius (r_+), for $\alpha' = 0.5$, $Q = 15$ and $b = 0.60$ at the critical point r_1.*

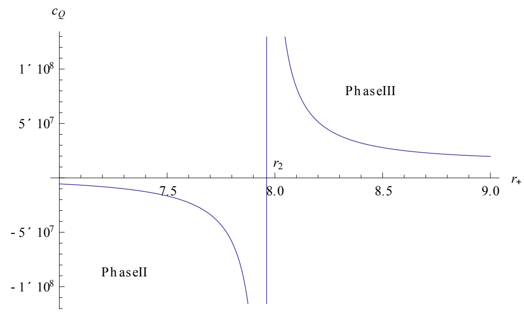

Figure 3.4: *Plot of specific heat (C_Q) against horizon radius (r_+), for $\alpha' = 0.5$, $Q = 15$ and $b = 0.60$ at the critical point r_2.*

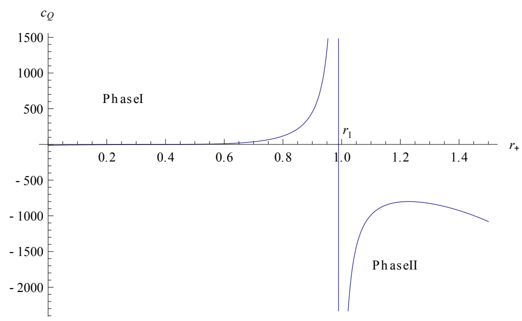

Figure 3.5: *Plot of specific heat (C_Q) against horizon radius (r_+), for $\alpha' = 0.5$, $Q = 0.50$ and $b = 10$ at the critical point r_1.*

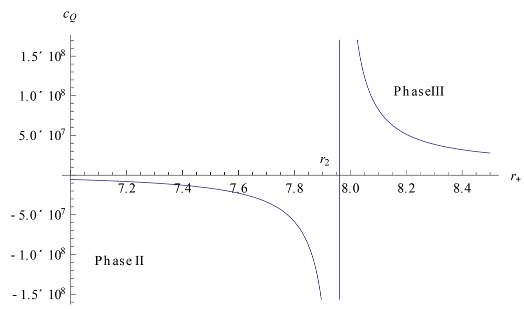

Figure 3.6: *Plot of specific heat (C_Q) against horizon radius (r_+), for $\alpha' = 0.5$, $Q = 0.50$ and $b = 10$ at the critical point r_2.*

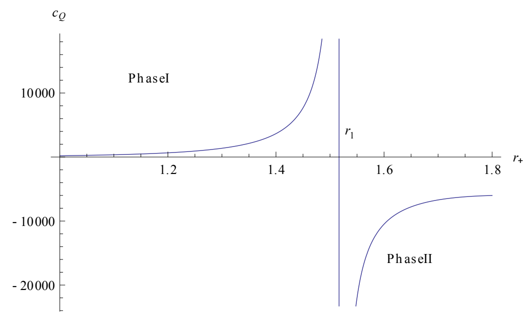

Figure 3.7: *Plot of specific heat* (C_Q) *against horizon radius* (r_+), *for* $\alpha' = 1.0$, $Q = 5$ *and* $b = 0.5$ *at the critical point* r_1.

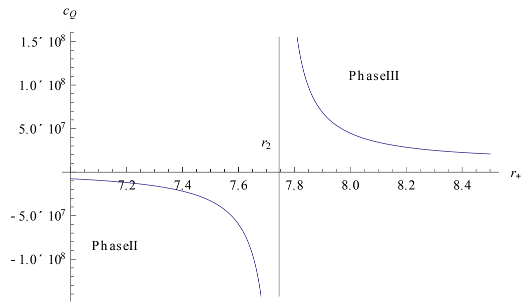

Figure 3.8: *Plot of specific heat* (C_Q) *against horizon radius* (r_+), *for* $\alpha' = 1.0$, $Q = 5$ *and* $b = 0.5$ *at the critical point* r_2.

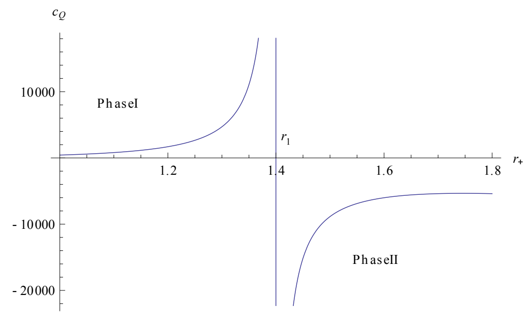

Figure 3.9: *Plot of specific heat* (C_Q) *against horizon radius* (r_+), *for* $\alpha' = 1.0$, $Q = 0.05$ *and* $b = 0.05$ *at the critical point* r_1.

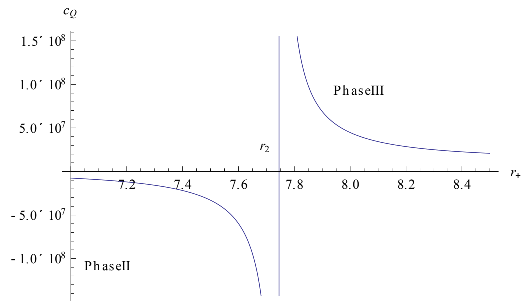

Figure 3.10: *Plot of specific heat* (C_Q) *against horizon radius* (r_+), *for* $\alpha' = 1.0$, $Q = 0.05$ *and* $b = 0.05$ *at the critical point* r_2.

The numerical values of the roots of Eq. (3.41) are given in Table 3.1a and Table 3.1b. For convenience we have written the real roots of Eq. (3.41) only. From our analysis it is observed that the specific heat always possesses simple poles. Moreover, there are two real positive roots ($r_i,\ i = 1, 2$) of the denominator of C_Q for different values of the parameters b, Q and α'. Also, from the $C_Q - r_+$ plots it is observed that the specific heat suffers discontinuity at the critical points r_i ($i = 1, 2$). This property of C_Q allows us to conclude that at the critical points there is indeed a continuous higher order phase transition[166, 167].

Let us now see whether there is any bound in the values of the parameters b, Q and α'. At this point it must be stressed that a bound in the parameter values (b, Q) for the Born-Infeld-AdS black holes in (3+1)-dimensions was found earlier[110, 166]. Interestingly, this bound is removed if we consider space-time dimensions greater than four[167]. Thus, it will be very much interesting to check whether the LBI-AdS black hole, which is indeed a $(6 + 1)$-dimensional black hole, possess similar features. In order to do so, we will consider the extremal LBI-AdS black hole. In this case both $f(r)$ and $\dfrac{df}{dr}$ vanish at the degenerate horizon r_e [110, 166, 167]. The

Table 3.1a: Real roots of Eq. (3.41) for $\alpha' = 0.5$ and $L = 10$

Q	b	r_1	r_2	r_3	r_4
15	0.6	1.59399	7.96087	-7.96087	-1.59399
8	0.2	1.18048	7.96088	-7.96088	-1.18048
5	0.5	1.25190	7.96088	-7.96088	-1.25190
0.8	20	1.01695	7.96088	-7.96088	-1.01695
0.3	15	0.975037	7.96088	-7.96088	-0.975037
0.5	10	0.989576	7.96088	-7.96088	-0.989576
0.5	1	0.989049	7.96088	-7.96088	-0.989049
0.5	0.5	0.987609	7.96088	-7.96088	-0.987609
0.05	0.05	0.965701	7.96088	-7.96088	-0.965701

Table 3.1b: Real roots of Eq. (3.41) for $\alpha' = 1.0$ and $L = 10$

Q	b	r_1	r_2	r_3	r_4
15	0.6	1.76824	7.74534	-7.74534	-1.76824
8	0.2	1.53178	7.74535	-7.74535	-1.53178
5	0.5	1.51668	7.74535	-7.74535	-1.51668
0.8	20	1.40553	7.74535	-7.74535	-1.40553
0.3	15	1.40071	7.74535	-7.74535	-1.40071
0.5	10	1.40214	7.74535	-7.74535	-1.40214
0.5	1	1.40214	7.74535	-7.74535	-1.40214
0.5	0.5	1.40213	7.74535	-7.74535	-1.40213
0.05	0.05	1.39992	7.74535	-7.74535	-1.39992

above two conditions for extremality result the following equation:

$$10r_e^4 + 5\alpha' r_e^2 + 2b^2 r_e^6 \left(1 - \sqrt{1 + \frac{16Q^2}{b^2\pi^4 r_e^{10}}}\right) - \Lambda r_e^6 = 0. \tag{3.44}$$

In Table 3.2a and 3.2b we give the numerical solutions of Eq. (3.44) for different choices of the values of the parameters b and Q for fixed values of α'. From these analysis we observe that for arbitrary choices of the parameters b and Q we always obtain atleast one real positive root of Eq. (3.44). This implies that there exists a smooth extremal limit for arbitrary b and Q, and there is no bound on the parameter space for a particular value of α'. Thus, the result obtained here (regarding the bound in the parameter values) is in good agreement with that obtained in Ref.[167].

Let us now analyze the phase structure and thermodynamic stability of the third order LBI-AdS black hole. If we proceed in the same line of analysis as of Chapter 2, the behavior of the specific heat (C_Q) at the critical points (Fig. 3.3-Fig. 3.10) classifies the following three phases of the black hole, namely, Phase I ($0 < r_+ < r_1$), Phase II ($r_1 < r_+ < r_2$) and Phase III ($r_+ > r_2$). Since the higher mass black hole possesses larger entropy/horizon radius, there is a phase transition

Table 3.2a: Roots of Eq. (3.44) for $\alpha' = 0.5$ and $L = 10$

Q	b	r_{e1}^2	r_{e2}^2	$r_{e3,e4}^2$	r_{e5}^2
15	0.6	-66.4157	-	-	+0.632734
8	0.2	-66.4157	-	-	+0.121590
5	0.5	-66.4157	-	-	+0.200244
0.8	20	-66.4157	-0.408088	-0.05881$\pm i$0.304681	+0.268514
0.3	15	-66.4157	-0.304225	-0.0517567$\pm i$0.175488	+0.145685
0.5	10	-66.4157	-0.349764	-0.0593449$\pm i$0.240144	+0.192519
0.5	1	-66.4157	-	-	+0.0221585
0.5	0.5	-66.4157	-	-	+0.00625335
0.05	0.05	-66.4157	-	-	$+6.57019\times10^{-7}$

Table 3.2b: Roots of Eq. (3.44) for $\alpha' = 1.0$ and $L = 10$

Q	b	r_{e1}^2	r_{e2}^2	$r_{e3,e4}^2$	r_{e5}^2
15	0.6	-66.1628	-	-	+0.50473
8	0.2	-66.1628	-	-	+0.0546589
5	0.5	-66.1628	-	-	+0.110054
0.8	20	-66.1629	-0.563564	-0.0913395$\pm i$0.266887	+0.236186
0.3	15	-66.1629	-0.514815	-0.0584676$\pm i$0.14609	+0.116834
0.5	10	-66.1629	-0.531635	-0.0760937$\pm i$0.210702	+0.155024
0.5	1	-66.1629	-0.542417	-	+0.00640486
0.5	0.5	-66.1629	-	-	+0.00163189
0.05	0.05	-66.1629	-	-	$+1.64256\times10^{-7}$

at r_1 from smaller mass black hole (Phase I) to intermediate (higher mass) black hole (Phase II). The critical point r_2 corresponds to a phase transition from an intermediate (higher mass) black hole (Phase II) to a larger mass black hole (Phase III). Moreover, from the $C_Q - r_+$ plots we note that the specific heat C_Q is positive for Phase I and Phase III whereas it is negative for Phase II. Therefore Phase I and Phase III correspond to thermodynamically stable phase ($C_Q > 0$) whereas Phase II corresponds to thermodynamically unstable phase ($C_Q < 0$).

We can further extend our stability analysis by considering the free energy of the LBI-AdS black hole. The free energy plays an important role in the theory of phase transition as well as in critical phenomena (Section 3.4). We may define the free energy of the third order LBI-AdS black hole as

$$\mathscr{F}(r_+, Q) = M(r_+, Q) - TS. \tag{3.45}$$

Using Eqs.(3.38), (3.39), and (3.40) we can write Eq. (3.45) as

$$\mathscr{F} = \frac{5\pi^2}{16}\left[\frac{\alpha'^2}{3} + r_+^4 + \alpha' r_+^2 + \frac{2b^2r_+^6}{15}\left(1 - \sqrt{1 + \frac{16Q^2}{b^2\pi^4 r_+^{10}}} - \frac{\Lambda}{2b^2} + \frac{20Q^2}{b^2\pi^4 r_+^{10}}\right.\right.$$
$$\left.\left.\mathscr{H}\left(\frac{1}{2}, \frac{2}{5}, \frac{7}{5}, -\frac{16Q^2}{b^2\pi^4 r_+^{10}}\right)\right)\right] - \frac{\pi^2\left(r_+^5 + \frac{10\alpha' r_+^3}{3} + 5\alpha'^2 r_+\right)}{40 r_+(r_+^2 + \alpha')^2}\left[10r_+^4 + 5\alpha' r_+^2\right.$$
$$\left. + 2b^2 r_+^6\left(1 - \sqrt{1 + \frac{16Q^2}{b^2\pi^4 r_+^{10}}}\right) - \Lambda r_+^6\right]. \tag{3.46}$$

In Figs. 3.11, 3.12, 3.13, and 3.14 we have given the plots of the free energy ($\mathscr{F}$) of the black hole with the radius of the outer horizon r_+. The free energy ($\mathscr{F}$) has a minima $\mathscr{F} = \mathscr{F}_m$ at $r_+ = r_m$. This point of minimum free energy is exactly the same as the first critical point $r_+ = r_1$ where the black hole shifts from a stable to an unstable phase. On the other hand $\mathscr{F}$ has a maxima $\mathscr{F} = \mathscr{F}_0$ at $r_+ = r_0$. The point at which $\mathscr{F}$ reaches its maximum value, is identical with the second critical point $r_+ = r_2$ where the black hole changes from unstable to stable phase. We can further divide the $\mathscr{F} - r_+$ plot into three distinct regions. In the first region $r_1' < r_+ < r_m$ the negative free energy decreases until it reaches the minimum value ($\mathscr{F}_m$) at $r_+ = r_m$. This region corresponds to the stable phase (Phase I: $C_Q > 0$) of the black hole. The free energy changes its slope at $r_+ = r_m$ and continues to increase in the second region $r_m < r_+ < r_0$ approaching towards the maximum value ($\mathscr{F}_0$) at $r_+ = r_0$. This region corresponds to the Phase II of the $C_Q - r_+$ plot where the black hole becomes unstable ($C_Q < 0$). The free energy changes its slope once again at $r_+ = r_0$ and decreases to zero at $r_+ = r_2'$ and finally becomes negative for $r_+ > r_2'$. This region of the $\mathscr{F} - r_+$ plot corresponds to the Phase III of the $C_Q - r_+$ plot where the black hole finally becomes stable ($C_Q > 0$).

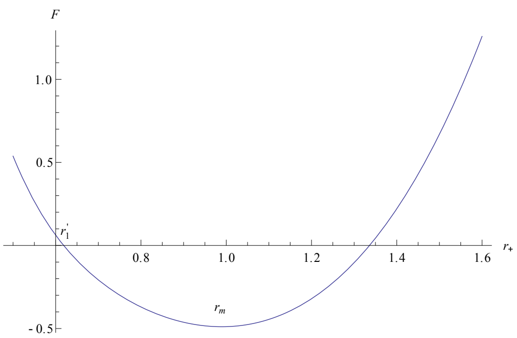

Figure 3.11: *Plot of free energy($\mathscr{F}$) against horizon radius (r_+), for $\alpha' = 0.50$, $Q = 0.50$ and $b = 10$.*

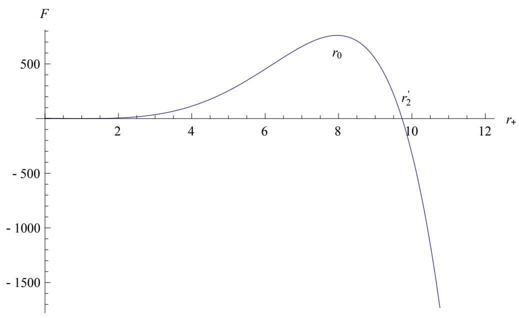

Figure 3.12: *Plot of free energy($\mathscr{F}$) against horizon radius (r_+), for $\alpha' = 0.50$, $Q = 0.50$ and $b = 10$.*

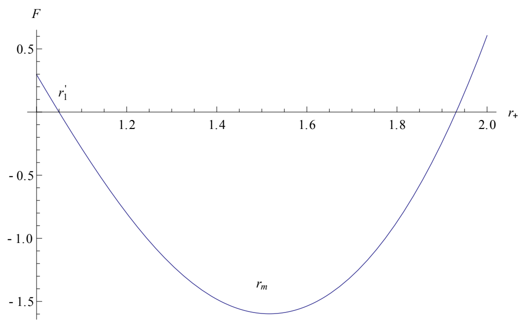

Figure 3.13: *Plot of free energy($\mathscr{F}$) against horizon radius (r_+), for $\alpha' = 1.0$, $Q = 5$ and $b = 0.5$.*

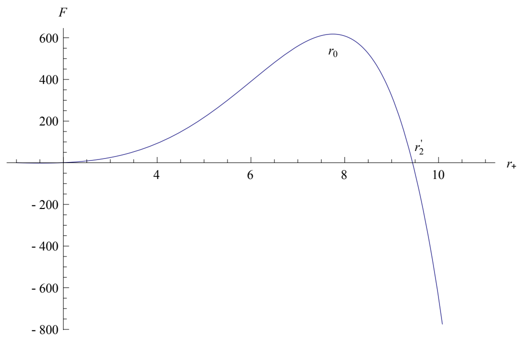

Figure 3.14: *Plot of free energy($\mathscr{F}$) against horizon radius (r_+), for $\alpha' = 1.0$, $Q = 5$ and $b = 0.5$.*

3.4 Critical exponents and scaling hypothesis

In ordinary thermodynamics, the theory of phase transition plays a crucial role to understand the behavior of a thermodynamic system. The behavior of thermodynamic quantities near the critical point(s) of phase transition gives a considerable amount of information about the system. The behavior of a thermodynamic system near the critical point(s) is usually studied by means of a set of indices known as the critical exponents[150, 151, 152]. These are generally denoted by a set of *Greek letters*: α, β, γ, δ, ϕ, ψ, η, ν. The critical exponents describe the nature of singularities in various measurable thermodynamic quantities near the critical point(s).

The sole purpose of the present section is to determine the first six static critical exponents (α, β, γ, δ, ϕ, ψ). For this purpose we shall follow the method discussed in Refs. [166]-[169]. Next, we discuss about the static scaling laws and static scaling hypothesis. Finally, we determine the other two critical exponents (ν and η) from two additional scaling laws.

3.4.1 Critical exponents

(1) ***Critical exponent*** α***:*** In order to determine the critical exponent α which is associated with the singularity of C_Q near the critical points r_i ($i = 1, 2$), we choose a point in the infinitesimal neighborhood of r_i as,

$$r_+ = r_i(1 + \Delta), \quad i = 1, 2, \tag{3.47}$$

where $|\Delta| << 1$. Let us denote the temperature at the critical point by $T(r_i)$ and define the quantity

$$\epsilon = \frac{T(r_+) - T(r_i)}{T(r_i)} \tag{3.48}$$

such that $|\epsilon| << 1$.

We now Taylor expand $T(r_+)$ in the neighborhood of r_i keeping the charge constant ($Q = Q_c$), which yields

$$T(r_+) = T(r_i) + \left[\left(\frac{\partial T}{\partial r_+}\right)_{Q=Q_c}\right]_{r_+=r_i}(r_+ - r_i) + \frac{1}{2}\left[\left(\frac{\partial^2 T}{\partial r_+^2}\right)_{Q=Q_c}\right]_{r_+=r_i}(r_+ - r_i)^2 + \cdots \tag{3.49}$$

Since the divergence of C_Q results from the vanishing of $\left(\frac{\partial T}{\partial r_+}\right)_Q$ at the critical point r_i (Eq. (3.41)), we may write Eq. (3.49) as

$$T(r_+) = T(r_i) + \frac{1}{2}\left[\left(\frac{\partial^2 T}{\partial r_+^2}\right)_{Q=Q_c}\right]_{r_+=r_i}(r_+ - r_i)^2, \tag{3.50}$$

where we have neglected the higher order terms in Eq. (3.49).

Using Eqs. (6.54) and (3.48) we can finally write Eq. (3.50) as

$$\Delta = \frac{\epsilon^{1/2}}{\Gamma_i^{1/2}}, \tag{3.51}$$

where

$$\Gamma_i = \frac{r_i^2}{2T(r_i)}\left[\left(\frac{\partial^2 T}{\partial r_+^2}\right)_{Q=Q_c}\right]_{r_+=r_i}. \tag{3.52}$$

The detailed expression of Γ_i is very much cumbersome and we shall not write it for the present work.

If we examine the $T - r_+$ plots (Fig. 3.1 & Fig. 3.2) we observe that, near the critical point $r_+ = r_1$ (which corresponds to the 'hump') $T(r_+) < T(r_1)$ so that $\epsilon < 0$, and on the contrary, near the critical point $r_+ = r_2$ (which corresponds to the 'dip') $T(r_+) > T(r_2)$ implying $\epsilon > 0$.

Substituting Eq. (3.47) into Eq. (3.41) we can write the singular part of C_Q as

$$C_Q = \frac{\mathscr{N}'(r_i, Q_c)}{\Delta \cdot \mathscr{D}'(r_i, Q_c)} \tag{3.53}$$

where $\mathscr{N}'(r_i, Q_c)$ is the value of the numerator of C_Q (Eq. (3.42)) at the critical point $r_+ = r_i$ and critical charge $Q = Q_c$. The expression for $\mathscr{D}'(r_i, Q_c)$ is given by

$$\mathscr{D}'(r_i, Q_c) = \mathscr{D}_1'(r_i, Q_c) + \mathscr{D}_2'(r_i, Q_c) + \mathscr{D}_3'(r_i, Q_c) \tag{3.54}$$

where

$$\mathscr{D}_1'(r_i, Q_c) = 10\pi^4 r_i^4\sqrt{1 + \frac{16Q_c^2}{b^2\pi^4 r_i^{10}}}\left[\left(2\alpha'^2 + 2b^2 r_i^6 + 8b^2 r_i^4\alpha'\right)\right.$$
$$\left. - \left(\Lambda r_i^6 + 9\alpha' r_i^2 + 4(2 + \Lambda\alpha')r_i^4\right)\right], \tag{3.55}$$

$$\mathscr{D}_2'(r_i, Q_c) = -\frac{80Q_c^2}{b^2 r_i^2}\left[\frac{15\alpha'}{r_i^2} + \frac{5\alpha'^2}{r_i^4} - \Lambda r_i^2 - 5(2+\Lambda\alpha')\right], \tag{3.56}$$

$$\mathscr{D}_3'(r_i, Q_c) = -20b^2\pi^4 r_i^8\left[r_i^2\left(1 + \frac{8Q_c^2}{b^2\pi^4 r_i^{10}}\right) + 4\alpha'\left(1 + \frac{10Q_c^2}{b^2\pi^4 r_i^{10}}\right)\right]. \tag{3.57}$$

It is to be noted that while expanding the denominator of C_Q we have retained the terms which are linear in Δ, and all other higher order terms of Δ have been neglected.

Using Eq. (3.53) we may summarize the critical behavior of C_Q near the critical points (r_1 and r_2) as follows:

$$C_Q \sim \begin{cases} \left[\dfrac{\mathscr{A}_i}{(-\epsilon)^{1/2}}\right]_{r_i=r_1} & \epsilon < 0 \\ \left[\dfrac{\mathscr{A}_i}{(+\epsilon)^{1/2}}\right]_{r_i=r_2} & \epsilon > 0, \end{cases} \tag{3.58}$$

where

$$\mathscr{A}_i = \frac{\Gamma_i^{1/2}\mathscr{N}'(r_i, Q_c)}{\mathscr{D}'(r_i, Q_c)}. \tag{3.59}$$

We can combine the rhs of Eq. (3.58) into a single expression, which describes the singular nature of C_Q near the critical point r_i, yielding

$$\begin{aligned} C_Q &= \frac{\mathscr{A}_i}{|\epsilon|^{1/2}} \\ &= \frac{\mathscr{A}_i T_i^{1/2}}{|T - T_i|^{1/2}}, \end{aligned} \tag{3.60}$$

where we have used Eq. (3.48). Here T and T_i are the abbreviations of $T(r_+)$ and $T(r_i)$, respectively.

We can now compare Eq. (3.60) with the standard form

$$C_Q \sim |T - T_i|^{-\alpha} \tag{3.61}$$

which gives $\alpha = \dfrac{1}{2}$.

(2) ***Critical exponent*** *β* ***:*** The critical exponent β is related to the electric potential at infinity (Φ) by the relation

$$\Phi(r_+) - \Phi(r_i) \sim |T - T_i|^{\beta}, \tag{3.62}$$

where the charge (Q) is kept constant.

Near the critical point $r_+ = r_i$ the Taylor expansion of $\Phi(r_+)$ yields

$$\Phi(r_+) = \Phi(r_i) + \left[\left(\frac{\partial\Phi}{\partial r_+}\right)_{Q=Q_c}\right]_{r_+=r_i} (r_+ - r_i) + \cdots \tag{3.63}$$

Neglecting the higher order terms and using Eqs. (3.30) and (3.51) we may rewrite Eq. (3.63) as,

$$\Phi(r_+) - \Phi(r_i) = -\left(\frac{4Q_c}{\pi^2 r_i^4 \Gamma_i^{1/2} T_i^{1/2}\sqrt{1+\frac{16Q_c^2}{b^2\pi^4 r_i^{10}}}}\right)|T - T_i|^{1/2}. \tag{3.64}$$

Comparing Eq. (3.64) with Eq. (3.62) we finally obtain $\beta = \dfrac{1}{2}$.

(3) ***Critical exponent*** γ***:*** Let us now determine the critical exponent γ which is associated with the singularity of the inverse of the isothermal compressibility (κ_T^{-1}) at constant charge $Q = Q_c$ near the critical point $r_+ = r_i$ as

$$\kappa_T^{-1} \sim |T - T_i|^{-\gamma}. \tag{3.65}$$

In order to calculate κ_T^{-1} we use the standard thermodynamic definition

$$\begin{aligned}\kappa_T^{-1} &= Q\left(\frac{\partial \Phi}{\partial Q}\right)_T \\ &= -Q\left(\frac{\partial \Phi}{\partial T}\right)_Q \left(\frac{\partial T}{\partial Q}\right)_\Phi\end{aligned} \tag{3.66}$$

where in the last line of Eq. (3.66) we have used the identity

$$\left(\frac{\partial \Phi}{\partial T}\right)_Q \left(\frac{\partial T}{\partial Q}\right)_\Phi \left(\frac{\partial Q}{\partial \Phi}\right)_T = -1. \tag{3.67}$$

Using Eqs. (3.30) and (3.39) we can write Eq. (3.66) as

$$\kappa_T^{-1} = \frac{\Omega(r_+, Q)}{\mathscr{D}(r_+, Q)}, \tag{3.68}$$

where $\mathscr{D}(r_+, Q)$ is the denominator identically equal to Eq. (3.43) (the denominator of C_Q) and the expression for $\Omega(r_+, Q)$ may be written as

$$\begin{aligned}\Omega(r_+, Q) = \frac{Q}{5\pi^2 r_+^4}\Bigg[& 128Q^2 + \left(15\pi^4 r_+^6\alpha + 5\pi^4 r_+^4\alpha^2 - \Lambda\pi^4 r_+^{10} - 5\pi^4 r_+^8(2+\alpha\Lambda)\right) \\ & \sqrt{1+\frac{16Q^2}{b^2\pi^4 r_+^{10}}} - \left(2b^2\pi^4 r_+^{10} + 10b^2\pi^4 r_+^8\alpha\right)\left(1 - \sqrt{1+\frac{16Q^2}{b^2\pi^4 r_+^{10}}}\right) \\ & \mathscr{H}\left(\frac{1}{2}, \frac{2}{5}, \frac{7}{5}, -\frac{16Q^2}{b^2\pi^4 r_+^{10}}\right)\Bigg] - \Sigma(r_+, Q),\end{aligned} \tag{3.69}$$

where

$$\Sigma(r_+,Q) = \frac{4Q}{5}\Bigg[2b^2\pi^2r_+^4\Big(r_+^2+5\alpha\Big)\sqrt{1+\frac{16Q^2}{b^2\pi^4r_+^{10}}}+\pi^2\Bigg(-15r_+^2\alpha-5\alpha^2+r_+^6\Big(\Lambda-2b^2\Big) + 5r_+^4\Big(2+\Lambda\alpha-2b^2\alpha\Big)\Bigg)\Bigg]. \tag{3.70}$$

From Eq. (3.68) we observe that κ_T^{-1} possesses simple poles. Moreover, κ_T^{-1} and C_Q exhibit common singularities.

Now, our interest is in the behavior of κ_T^{-1} near the critical point $r_+ = r_i$. In order to do so we substitute Eq. (3.47) into Eq. (3.68). The resulting equation for the singular part of κ_T^{-1} may be written as,

$$\kappa_T^{-1} = \frac{\Omega'(r_i,Q_c)}{\Delta\cdot\mathscr{D}'(r_i,Q_c)}. \tag{3.71}$$

In Eq. (3.71), $\Omega'(r_i,Q_c)$ is the value of the numerator of κ_T^{-1} (Eq. (3.69)) at the critical point $r_+ = r_i$ and critical charge $Q = Q_c$, whereas $\mathscr{D}'(r_i,Q_c)$ was identified earlier as Eq. (3.54).

Substituting Eq. (3.51) into Eq. (3.71) we may express the singular nature of κ_T^{-1} near the critical points (r_1 and r_2) as

$$\kappa_T^{-1} \simeq \begin{cases} \left[\dfrac{\mathscr{B}_i}{(-\epsilon)^{1/2}}\right]_{r_i=r_1} & \epsilon<0 \\ \left[\dfrac{\mathscr{B}_i}{(+\epsilon)^{1/2}}\right]_{r_i=r_2} & \epsilon>0, \end{cases} \tag{3.72}$$

where

$$\mathscr{B}_i = \frac{\Gamma_i^{1/2}\Omega'(r_i,Q_c)}{\mathscr{D}'(r_i,Q_c)}.. \tag{3.73}$$

Combining the rhs of Eq. (3.72) into a single expression as before, we can express the singular behavior of κ_T^{-1} near the critical point r_i as

$$\begin{aligned}\kappa_T^{-1} &= \frac{\mathscr{B}_i}{|\epsilon|^{1/2}} \\ &= \frac{\mathscr{B}_iT_i^{1/2}}{|T-T_i|^{1/2}}.\end{aligned} \tag{3.74}$$

Comparing Eq. (3.74) with Eq. (3.65) we find $\gamma = \frac{1}{2}$.

(4) ***Critical exponent*** δ***:*** The critical exponent δ is associated with the electrostatic potential (Φ) for the fixed value $T = T_i$ of temperature. The relation can be written as

$$\Phi(r_+) - \Phi(r_i) \sim |Q-Q_i|^{1/\delta}. \tag{3.75}$$

In this relation Q_i is the value of charge (Q) at the critical point r_i. In order to obtain δ we first Taylor expand $Q(r_+)$ around the critical point $r_+ = r_i$. This yields

$$Q(r_+) = Q(r_i) + \left[\left(\frac{\partial Q}{\partial r_+}\right)_{T=T_i}\right]_{r_+=r_i}(r_+ - r_i) + \frac{1}{2}\left[\left(\frac{\partial^2 Q}{\partial r_+^2}\right)_{T=T_i}\right]_{r_+=r_i}(r_+ - r_i)^2 + ... \tag{3.76}$$

Neglecting the higher order terms we can write Eq. (3.76) as

$$Q(r_+) - Q(r_i) = \frac{1}{2}\left[\left(\frac{\partial^2 Q}{\partial r_+^2}\right)_T\right]_{r_+=r_i}(r_+ - r_i)^2. \tag{3.77}$$

Note that, here we have used the standard thermodynamic identity

$$\left[\left(\frac{\partial Q}{\partial r_+}\right)_T\right]_{r_+=r_i}\left[\left(\frac{\partial r_+}{\partial T}\right)_Q\right]_{r_+=r_i}\left(\frac{\partial T}{\partial Q}\right)_{r_+=r_i} = -1 \tag{3.78}$$

and considered the fact that at the critical point $r_+ = r_i$, $\left(\frac{\partial T}{\partial r_+}\right)_Q$ vanishes.

Let us now define a quantity

$$\Upsilon = \frac{Q(r_+) - Q_i}{Q_i} = \frac{Q - Q_i}{Q_i}, \tag{3.79}$$

where $|\Upsilon| << 1$. Here we denote $Q(r_+)$ and $Q(r_i)$ by Q and Q_i, respectively.

Using Eqs. (3.47) and (3.79) we obtain from Eq. (3.77)

$$\Delta = \frac{\Upsilon^{1/2}}{\Psi_i^{1/2}}\left[\frac{2Q_i}{r_i^2}\right]^{1/2}, \tag{3.80}$$

where

$$\Psi_i = \left[\left(\frac{\partial^2 Q}{\partial r_+^2}\right)_T\right]_{r_+=r_i}. \tag{3.81}$$

The expression for Ψ_i is very much cumbersome, and we shall not write it here. We now consider the functional relation

$$\Phi = \Phi(r_+, Q) \tag{3.82}$$

from which we may write

$$\left[\left(\frac{\partial \Phi}{\partial r_+}\right)_T\right]_{r_+=r_i} = \left[\left(\frac{\partial \Phi}{\partial r_+}\right)_Q\right]_{r_+=r_i} + \left[\left(\frac{\partial Q}{\partial r_+}\right)_T\right]_{r_+=r_i}\left(\frac{\partial \Phi}{\partial Q}\right)_{r_+=r_i}. \tag{3.83}$$

Using Eq. (3.78) we can rewrite Eq. (3.83) as

$$\left[\left(\frac{\partial \Phi}{\partial r_+}\right)_{T=T_i}\right]_{r_+=r_i} = \left[\left(\frac{\partial \Phi}{\partial r_+}\right)_{Q=Q_c}\right]_{r_+=r_i}. \tag{3.84}$$

Now the Taylor expansion of Φ at constant temperature around $r_+ = r_i$ yields

$$\Phi(r_+) = \Phi(r_i) + \left[\left(\frac{\partial \Phi}{\partial r_+}\right)_{T=T_i}\right]_{r_+=r_i} (r_+ - r_i), \tag{3.85}$$

where we have neglected all the higher order terms.

Finally using Eqs.(3.80), (3.84), and (3.30) we may write Eq. (3.85) as,

$$\Phi(r_+) - \Phi(r_i) = \left(\frac{-4Q_c}{\pi^2 r_i^5 \sqrt{1 + \frac{16Q_c^2}{b^2\pi^4 r_i^{10}}}}\right) \left(\frac{2}{\Psi_i}\right)^{\frac{1}{2}} |Q - Q_i|^{\frac{1}{2}}. \tag{3.86}$$

Comparing Eqs. (3.75) and (3.86) we find that $\delta = 2$.

(5) ***Critical exponent*** ϕ***:*** The critical exponent ϕ is associated with the divergence of the specific heat at constant charge (C_Q) at the critical point $r_+ = r_i$ as

$$C_Q \sim |Q - Q_i|^{-\phi}. \tag{3.87}$$

From eqs.(3.53) and (3.60) we note that

$$C_Q \sim \frac{1}{\Delta} \tag{3.88}$$

which may further be written as,

$$C_Q \sim \frac{1}{|Q - Q_i|^{1/2}}, \tag{3.89}$$

where we have used Eq. (3.80).

Comparison of Eq. (3.89) with Eq. (3.87) yields $\phi = \frac{1}{2}$.

(6) ***Critical exponent*** ψ***:*** In order to calculate the critical exponent ψ, which is related to the entropy of the third order LBI-AdS black hole, we Taylor expand the entropy ($S(r_+)$) around the critical point $r_+ = r_i$. This gives

$$S(r_+) = S(r_i) + \left[\left(\frac{\partial S}{\partial r_+}\right)\right]_{r_+=r_i} (r_+ - r_i) + \cdots \tag{3.90}$$

If we now neglect all the higher order terms and use eqs.(3.40), (3.47), and (3.80), we can write Eq. (3.90) as

$$S(r_+) - S(r_i) = \frac{5\pi^3}{4}\left(r_i^4 + 2\alpha' r_i^2 + \alpha'^2\right)\left(\frac{2}{\Psi_i}\right)^{1/2} |Q - Q_i|^{1/2}. \tag{3.91}$$

Comparing Eq. (3.91) with the standard relation

$$S(r_+) - S(r_i) \sim |Q - Q_c|^{\psi}, \tag{3.92}$$

we finally obtain $\psi = \frac{1}{2}$.

In the following Table 3.3 we write all the six critical exponents obtained from our analysis in a tabular form. For comparison we also give the critical exponents associated with some well known systems.

Table 3.3: Critical exponents of different systems

Critical exponents	LBI-AdS black hole	$CrBr_3$ *	2D Ising model*	van der Waals's system‡
α	0.5	0.05	0	0
β	0.5	0.368	0.125	0.5
γ	0.5	1.215	1.7	1.0
δ	2.0	4.28	15	3.0
ψ	0.5	0.60	-	-
ϕ	0.5	0.03	-	-

(*: these are the non-mean field values.)
(‡: these values are taken from Ref. [163].)

3.4.2 Scaling laws and scaling hypothesis

In order to complete our study of critical phenomena in LBI-AdS black hole, we discuss the *thermodynamic scaling laws* and *static scaling hypothesis* for this black hole. In standard thermodynamic systems the critical exponents are found to satisfy some relations among themselves. These relations are called *thermodynamic scaling laws* [150]-[152]. These scaling relations are written as

$$\begin{aligned}
\alpha + 2\beta + \gamma &= 2 \\
\alpha + \beta(\delta + 1) &= 2 \\
\phi + 2\psi - \frac{1}{\delta} &= 1 \\
\beta(\delta - 1) &= \gamma \\
(2 - \alpha)(\delta - 1) &= \gamma(1 + \delta) \\
1 + (2 - \alpha)(\delta\psi - 1) &= (1 - \alpha)\delta.
\end{aligned} \tag{3.93}$$

From the values of the critical exponents obtained in our analysis (see Section 3.4.1, Table 3.3) it is easy to check that these scaling relations (Eq. (3.93)) are indeed satisfied for the LBI-AdS black hole.

We are now in a position to explore the *static scaling hypothesis* [151]-[153] for this black hole. Since we are working in the *canonical ensemble* framework, the thermodynamic potential of interest is the *Helmhotz free energy*, $\mathscr{F}(T, Q) = M - TS$, where the symbols have their usual meaning.

Now the static scaling hypothesis states that, *Close to the critical point the singular part of the Helmhotz free energy is a generalized homogeneous function of its variables [150]-[152].*

This asserts that there exist two parameters a_ϵ and a_Υ such that

$$\mathscr{F}(\lambda^{a_\epsilon}\epsilon, \lambda^{a_\Upsilon}\Upsilon) = \lambda\mathscr{F}(\epsilon, \Upsilon) \tag{3.94}$$

for any arbitrary number λ.

In an attempt to find the values of the scaling parameters a_ϵ and a_Υ, we Taylor expand the Helmhotz free energy $\mathscr{F}(T, Q)$ near the critical point $r_+ = r_i$. The result is the following:

$$\begin{aligned}\mathscr{F}(T,Q) = \mathscr{F}(T,Q)|_{r_+=r_i} &+ \left[\left(\frac{\partial \mathscr{F}}{\partial T}\right)_Q\right]_{r_+=r_i}(T-T_i) + \frac{1}{2}\left[\left(\frac{\partial^2 \mathscr{F}}{\partial T^2}\right)_Q\right]_{r_+=r_i}(T-T_i)^2 \\ &+ \left[\left(\frac{\partial \mathscr{F}}{\partial Q}\right)_T\right]_{r_+=r_i}(Q-Q_i) + \frac{1}{2}\left[\left(\frac{\partial^2 \mathscr{F}}{\partial Q^2}\right)_T\right]_{r_+=r_i}(Q-Q_i)^2 \\ &+ \left[\left(\frac{\partial^2 \mathscr{F}}{\partial T \partial Q}\right)\right]_{r_+=r_i}(T-T_i)(Q-Q_i) + \cdots .\end{aligned} \tag{3.95}$$

From Eq. (3.95) we can identify the second derivatives of $\mathscr{F}$ as

$$\left(\frac{\partial^2 \mathscr{F}}{\partial T^2}\right)_Q = \frac{-C_Q}{T}, \tag{3.96}$$

$$\left(\frac{\partial^2 \mathscr{F}}{\partial Q^2}\right)_T = \frac{\kappa_T^{-1}}{Q}. \tag{3.97}$$

Since both C_Q and κ_T^{-1} diverge at the critical point, these derivatives can be justified as the singular parts of the free energy $\mathscr{F}$.

Now, in the theory of critical phenomena we are mainly interested in the singular part of the relevant thermodynamic quantities. Therefore, we sort out the singular part of $\mathscr{F}(T, Q)$ from Eq. (3.95), which may be written as

$$\begin{aligned}\mathscr{F}_s &= \frac{1}{2}\left[\left(\frac{\partial^2 \mathscr{F}}{\partial T^2}\right)_Q\right]_{r_+=r_i}(T-T_i)^2 + \frac{1}{2}\left[\left(\frac{\partial^2 \mathscr{F}}{\partial Q^2}\right)_T\right]_{r_+=r_i}(Q-Q_i)^2 \\ &= \frac{-C_Q}{2T_i}(T-T_i)^2 + \frac{\kappa_T^{-1}}{2Q_i}(Q-Q_i)^2,\end{aligned} \tag{3.98}$$

where the subscript "s" denotes the singular part of the free energy $\mathscr{F}$.

Using Eqs. (3.51), (3.60), (3.74), and (3.80) we may write the singular part of the Helmhotz free energy ($\mathscr{F}$) as

$$\mathscr{F}_s = \sigma_i \epsilon^{3/2} + \tau_i \Upsilon^{3/2} \tag{3.99}$$

where

$$\sigma_i = \frac{-\mathscr{A}_i T_i}{2} \quad \text{and} \quad \tau_i = \frac{\mathscr{B}_i \Psi_i^{1/2} Q_i^{1/2} r_i}{2^{3/2}\Gamma_i^{1/2}}. \tag{3.100}$$

From Eqs. (3.94) and (3.99) we observe that

$$a_\epsilon = a_\Upsilon = \frac{2}{3}. \tag{3.101}$$

This is an interesting result in the sense that, in general, a_ϵ and a_Υ are different for a generalized homogeneous function (GHF), but in this particular model of the black hole these two scaling parameters are indeed identical. With this result in hand we can argue that the Helmhotz free energy is an usual homogeneous function for the third order LBI-AdS black hole. Moreover, we can determine the critical exponents (α, β, γ, δ, ϕ, ψ) once we calculate the scaling parameters. This is because these critical exponents are related to the scaling parameters as[151, 152]

$$\alpha = 2 - \frac{1}{a_\epsilon}, \qquad \beta = \frac{1 - a_\Upsilon}{a_\epsilon},$$
$$\gamma = \frac{2a_\Upsilon - 1}{a_\epsilon}, \qquad \delta = \frac{a_\Upsilon}{1a_\Upsilon},$$
$$\phi = \frac{2a_\epsilon - 1}{a_\Upsilon}, \qquad \psi = \frac{1 - a_\epsilon}{a_\Upsilon}. \tag{3.102}$$

3.4.3 Additional exponents

There are two other critical exponents associated with the behavior of the *correlation function* and *correlation length* of the system near the critical surface. We shall denote these two critical exponents as η and ν, respectively. If $G(\vec{r}_+)$ and ξ are the correlation function and the correlation length we can relate η and ν with them as

$$G(\vec{r}_+) \sim r_+^{2-n-\eta} \tag{3.103}$$

and

$$\xi \sim |T - T_i|^{-\nu}. \tag{3.104}$$

For the time being we shall assume that the two additional scaling relations[151]

$$\gamma = \nu(2 - \eta) \quad \text{and} \quad (2 - \alpha) = \nu n \tag{3.105}$$

hold for the third order LBI-AdS black hole. Using these two relations (Eq. (3.105)) and the values of α and γ, the exponents ν and η are found to be 1/4 and 0, respectively.

Although we have calculated η and ν assuming the additional scaling relations to be valid, it is not proven yet that these scaling relations are indeed valid for the black holes. One may adapt different techniques to calculate η and ν, but till now no considerable amount of progress has been made in this direction. One may compute these two exponents directly from the correlation of scalar modes in the theory of gravitation[165], but the present theories of critical phenomena in black holes are far from complete.

3.5 Conclusive remarks

In this chapter we have analyzed the critical phenomena in higher curvature charged black holes in a canonical framework. For this purpose we have considered the third order Lovelock-Born-Infeld-AdS (LBI-AdS) black holes in a spherically symmetric space-time. We systematically derived the thermodynamic quantities for such black holes. We have been able to show that some of the thermodynamic quantities (C_Q, κ_T^{-1}) diverge at the critical points. From the nature of the plots we have argued that there is a higher order phase transition in this black hole. Although the analytic estimation of the critical points has not been possible due to the complexity of the relevant equations, we have determined the critical points numerically. However, all the critical exponents have been calculated analytically near the critical points. Unlike the AdS black holes in the Einstein gravity, one interesting property of the higher curvature black holes is that the usual area law of entropy does not hold for these black holes. One might then expect that the critical exponents may differ form those for the AdS black holes in the Einstein gravity. But we have found that all the critical exponents in the third order LBI-AdS black hole are indeed identical with those obtained in Einstein as well as Horava-Lifshitz gravity[166, 167, 168]. From this observation we may conclude that these black holes belong to the same universality class. Moreover, the critical exponents take the mean field values. It is to be noted that these black holes have distinct set of critical exponents which does not match with the critical exponents of any other known thermodynamic systems. Another point that must be stressed is that the static critical exponents are independent of the spatial dimensionality of the AdS space-time. This suggests the mean field behavior in black holes as thermodynamic systems and allows us to study the phase transition phenomena in the black holes. We have also discussed the *static scaling laws* and *static scaling hypothesis*. The static critical exponents are found to satisfy the *static scaling laws* near the critical points. We have checked the consistency of the *static scaling hypothesis*. Apart from this, we note that the two scaling parameters have identical values. This allows us to conclude that the Helmhotz free energy is indeed a homogeneous function for this type of black hole. We have determined the two other critical exponents ν and η associated with the correlation length (ξ) and correlation function ($G(\vec{r}_+)$) near the critical surface assuming the validity of the additional scaling laws. The values of these two exponents are found to be $\frac{1}{4}$ and 0, respectively, in the six spatial dimensions. Although the other six critical exponents are independent of the spatial dimension of the system, these two exponents are very much dimension dependent.

In our analysis we have been able to resolve a number of vexing issues concerning the critical phenomena in LBI-AdS black holes. But there still remains some unsolved problems that lead us to make the following comments: *First of all*, we have made a qualitative argument about the nature of the phase transition in this black hole. One needs to go through detailed algebraic analysis in order to determine the true order of the phase transition[115]-[120]. *Secondly*, we have calculated the values of the exponents ν and η assuming that the additional scaling relations hold for this black hole. But there is no evidence whether these two laws hold for the black hole[166, 167, 168]. These scaling relations may or may not hold for the black hole. The dimension dependence of these two exponents (ν and η) makes the issue highly nontrivial in higher dimensions. A further attempt to determine η and ν may be based on Ruppeiner's prescription [129]-[135], where it is assumed that the absolute value of the thermodynamic scalar curvature ($|\mathscr{R}|$) is proportional to the correlation volume ξ^n:

$$|\mathscr{R}| \sim \xi^n \tag{3.106}$$

where n is the spatial dimension of the black hole. Now if we can calculate $\mathscr{R}$ using the standard method[120],[129]-[135], we can easily determine ξ from Eq. (3.106). Evaluating ξ around the critical point $r_+ = r_i$ as before, we can determine ν directly. It is then straight forward to calculate η by using (80). This alternative approach, based on Ruppeiner's prescription, to determine ν and η needs high mathematical rigor and also the complexity in the determination of the scalar curvature ($\mathscr{R}$) in higher dimensions makes the issue even more challenging.

At this point we must mention that the studies of non-linear aspects of black holes are not restricted within the context of classical gravity only, rather, it can further be extended by considering examples from sectors which are relatively new in modern high energy physics in which black holes are of significant importance in order to answer several questions that are still enigmatic. The AdS/CFT correspondence is one of them. This can describe aspects of certain strongly coupled field theories much of which was not understood well due to the lack of theoretical tools. In the remaining three chapters of the thesis we shall explore the non-linear aspects of s-wave holographic superconductors which are one of the striking predictions of the AdS/CFT dualities.

Chapter 4

Holographic s-wave Superconductors with Born-Infeld Correction

4.1 Overview

Superconductivity is one of the most surprising phenomena that has been observed in modern physics. Since its discovery in 1911, it has been a promising direction of research. In order to explain superconductivity Ginzburg and Landau proposed a phenomenological model in which superconductivity was explained in terms of a second order phase transition[201]. Latter on, in 1957, a microscopic theory of superconductivity was put forward by Bardeen, Cooper and Schrieffer which was found to be the exact description of weakly coupled superconductors[202]. However, the discovery of high-T_c superconductors has imposed doubt on the generality of the BCS theory for these are strongly interacting systems (e.g., cuprates, iron pnictides, etc.), and the mechanism responsible for the high-T_c superconductivity can not be explained in the framework of the BCS theory at all. Despite several attempts an appreciable microscopic theory of high-T_c superconductors is still intangible.

However, very recently an attempt has been made in this direction in the framework of the AdS/CFT duality[191]-[197]. This is encouraged by the fact that the AdS/CFT correspondence can be used as a theoretical tool to study strongly coupled systems[80],[170]-[175]. This conjecture states that a gravity theory in $(n+1)$-dimensional AdS space-time is dual to a quantum field theory residing on its n-dimensional boundary. According to Ref.[80], it is possible to find a mapping between the phase structure of AdS black holes in the bulk gravitational theory to that with the dual CFTs living on the boundary of the AdS. In this language, the problem of solving a strongly interacting field theory casts into the problem of finding solutions of the equations of motion of classical gravity in the bulk space-time subjected to some restrictions to certain parameters on both sides. Using this holographic framework, it has been possible to gain crucial theoretical insight into high-T_c superconductivity, hence justifying the name *holographic superconductors* (henceforth

HS). The basic mechanism that leads to superconductivity in these holographic models is discussed in details in Chapter 1.

Starting with the pioneering work of Hartnoll, Herzog and Holowitz[193], there has been a spate of papers in the literature describing various holographic models of superconductors[194]-[197],[206]-[240],[245]-[258],[276]-[290]. Although most of these holographic models are based on the usual framework of Maxwell electrodynamics [193]-[197],[206]-[223],[245]-[258],[276]-[290], studying these models in the framework of non-linear electrodynamics is an equally important direction of research[224]-[240].

In this chapter we focus our attention on several important aspects of HS in the presence of a higher derivative correction to the usual Maxwell electrodynamics, namely, the Born-Infeld correction. The importance and regime of applicability of the Born-Infeld theory are discussed in Chapter 1. Here, based on Ref.[231], we compute the critical temperature and the order parameter of condensation of HS in the presence of Born-Infeld corrections in the usual Einstein-Maxwell gravity. According to the holographic conjecture[193, 241], the scalar field which is coupled to the gravity action in the bulk allows either of its boundary expansion coefficients to act as order parameter for condensation in the boundary theory provided the mass-squared of the scalar field lies between a particular range[242]. Following this, we choose to carry out our analysis by considering the *sub-leading order boundary expansion coefficient* of the scalar field as the order parameter (see Eq. (4.12), Section 4.2). Note that, in Ref.[228] the analysis was performed by considering the leading order coefficient as the order parameter. There the analysis was not complete since the present boundary condition was not considered. In fact, the analytic computations of the mentioned physical quantities become exceedingly difficult once we take into account the present boundary condition. In our analysis we circumvent this difficulty by adopting a suitable analytic scheme. We observe that a holographic superconducting phase transition indeed takes place below certain critical temperature (T_c). We also establish a relation between charge density and critical temperature which is seen to be affected by the non-linear Born-Infeld parameter (b). Proceeding further, we derive the order parameter of condensation which is non-vanishing only when $T < T_c$. The explicit dependence of the order parameter on b is also exploited. Moreover, the results that we obtain are found to be in good agreement with the numerical results found earlier in Ref.[224].

Before presenting our mathematical analysis we would like to mention the sectioning of the present chapter. In Section 4.2 we present the minimal ingredients needed to construct the holographic superconductor. In Section 4.3 and Section 4.4 we compute the critical temperature and the order parameter for the condensation, respectively. Finally, we make some comments about our holographic model in Section 4.5.

4.2 Ingredients to construct holographic superconductors

In order to construct the holographic model of a s-wave superconductor we shall choose Schwarzschild-AdS space-time as our fixed background. Following the prescription of Refs.[193, 194], we shall consider a planar Schwarzschild-AdS black hole in this bulk space-time with the following metric:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(dx^2 + dy^2) \tag{4.1}$$

where the metric function can be written as

$$f(r) = \left(r^2 - \frac{r_+^3}{r}\right), \tag{4.2}$$

r_+ being the horizon radius of the black hole. Note that, we have set the radius of the bulk AdS space-time to unity ($L = 1$) which follows from the scaling properties of the equations of motion[220]. Also, we work in the system of natural units: $G = c = \hbar = k_B = 1$.
Given this metric Eq.(4.1), it is easy to obtain the Hawking temperature of the black hole by analytically continuing the metric to the Euclidean sector which comes out to be

$$T = \frac{3r_+}{4\pi}. \tag{4.3}$$

This in indeed the temperature of the field theory residing on the boundary of the AdS space. In accordance with Refs.[191, 192], we introduce a $U(1)$ gauge field and a charged massive complex scalar field in this fixed background in the form of an Abelian-Higgs action written as

$$\mathscr{L} = \mathscr{L}_{BI} - |\partial_\mu\psi - iqA_\mu\psi|^2 - m^2|\psi|^2. \tag{4.4}$$

In Eq.(4.4) $\mathscr{L}_{BI}$ is the Born-Infeld Lagrangian density given by[224]

$$\mathscr{L}_{BI} = \frac{1}{b}\left(1 - \sqrt{1 + \frac{bF}{2}}\right), \tag{4.5}$$

where $F \equiv F^{\mu\nu}F_{\mu\nu}$ and b is the Born-Infeld parameter. In the limit $b \to 0$ the Born-Infeld Lagrangian turns into the standard Maxwell Lagrangian. The equation of motion for the electromagnetic field tensor $F_{\mu\nu}$ can be written as

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\frac{\sqrt{-g}F^{\mu\nu}}{\sqrt{1 + \frac{bF}{2}}}\right) = i\left(\psi^*\partial^\nu\psi - \psi(\partial^\nu\psi)^*\right) + 2A^\nu|\psi|^2. \tag{4.6}$$

The equations of motion for the gauge field and the scalar field are non-linear differential equations which are solved by appropriate ansatz. In our work, we choose the planar symmetric ansatz $\psi = \psi(r)$ and $A_\mu = (\phi(r), 0, 0, 0)$[193, 194]. This leads to the following two coupled non-linear differential equations for the scalar and the gauge fields:

$$\psi''(r) + \left(\frac{f'}{f} + \frac{2}{r}\right)\psi'(r) + \left(\frac{\phi^2(r)}{f^2} + \frac{2}{f}\right)\psi(r) = 0, \tag{4.7}$$

$$\phi''(r) + \frac{2}{r}\left(1 - b\phi'^2(r)\right)\phi'(r) - \frac{2\psi^2(r)}{f}\left(1 - b\phi'^2(r)\right)^{\frac{3}{2}} = 0. \tag{4.8}$$

Note that, the above set of equations (Eqs. (4.7), (4.8)) are written in the radial coordinate r. In order to carry out an analytic computation, it is convenient to define a new variable $z = \frac{r_+}{r}$ which redefines the positions of the horizon and the boundary at $z = 1$ and $z = 0$, respectively. In this new variable Eq.(4.7) and Eq.(4.8) become

$$z\psi''(z) - \frac{2 + z^3}{1 - z^3}\psi'(z) + \left[z\frac{\phi^2(z)}{r_+^2(1 - z^3)^2} + \frac{2}{z(1 - z^3)}\right]\psi(z) = 0, \tag{4.9}$$

$$\phi''(z) + \frac{2bz^3}{r_+^2}\phi'^3(z) - \frac{2\psi^2(z)}{z^2(1 - z^3)}\left(1 - \frac{bz^4}{r_+^2}\phi'^2(z)\right)^{\frac{3}{2}}\phi(z) = 0. \tag{4.10}$$

At this point of discussions we must mention a few important points regarding our analysis, namely, *(i)* we have performed our entire analysis in the probe limit, *(ii)* we have investigated the effect of the higher derivative corrections to the gauge field in the leading order, i.e., we have kept terms only linear in b. Thus, the results derived from our analysis are valid only in the leading order of b, *(iii)* we have chosen the mass square of the scalar field (ψ) as $m^2 = -2$ which is well above the Breitenlohner-Freedman (BF) bound[242], *(iv)* in the field ansatz, the choices of the fields are justified since it is seen that under the transformations $A_\mu \to A_\mu + \partial_\mu\theta$ and $\psi \to \psi e^{i\theta}$ the above equations of motion remain invariant. This demands that the phase of ψ remains constant and we may take ψ to be real without any loss of generality.

In order to solve the above equations of motion (Eqs. (4.9),(4.10)) the following boundary conditions are considered[193, 194]:

- The regularity of the gauge field and the scalar field at the horizon ($z = 1$) implies

  $\phi(z = 1) = 0, \qquad \text{and} \qquad \psi(z = 1) = \frac{3}{2}\psi'(z = 1).$

- At the boundary of the AdS space-time ($z = 0$)

$$\phi(z) \approx \mu - \frac{\rho}{r} = \mu - \frac{\rho}{r_+}z \tag{4.11}$$

  and

$$\psi(z) \approx \frac{\psi^{(-)}}{r^{\Delta_-}} + \frac{\psi^{(+)}}{r^{\Delta_+}} = \frac{\psi^{(-)}}{r_+^{\Delta_-}}z^{\Delta_-} + \frac{\psi^{(+)}}{r_+^{\Delta_+}}z^{\Delta_+} \tag{4.12}$$

where $\Delta_\pm = \frac{3}{2} \pm \sqrt{\left(\frac{9}{4} + m^2\right)}$ is the conformal dimension of the *condensation operator* $\mathscr{O}$ in the boundary field theory, and μ and ρ are interpreted as the chemical potential and charge density of the dual field theory, respectively. Since we have considered $m^2 < 0$, we are left with two different condensation operators of different dimensionality corresponding to the choice of quantization of the scalar field ψ in the bulk[241]. In the present context either $\psi^{(+)}$ or $\psi^{(-)}$ will act as a condensation operator while the other will act as a source. In the present work we choose $\psi^{(+)} = \langle\mathscr{O}\rangle$ and $\psi^{(-)}$ as its source. Since we want the condensation to take place in the absence of any source, we set $\psi^{(-)} = 0$.

At this point, it must be stressed that for the present choice of ψ the analytic calculations of various entities near the critical point get notoriously difficult and special care should be taken in order to carry out a perturbative analysis. In the present work we focus to evade the above mentioned difficulties by adopting certain mathematical techniques. Our analysis indeed shows a good agreement with the numerical results existing in the literature[224].

4.3 Critical temperature for condensation

The presence of non-extremal black hole in the bulk places the system at finite temperature. In addition, the charge of the black hole gives rise to a chemical potential in the boundary field theory which subsequently introduces a new scale in the dual theory making a phase transition possible at a critical temperature, T_c.

At the critical temperature T_c the scalar field $\psi(z)$ vanishes, so Eq.(4.10) becomes

$$\phi''(z) + \frac{2bz^3}{r^2_{+(c)}}\phi'^3(z) = 0. \tag{4.13}$$

The solution of this non-trivial equation in the interval $[z, 1]$ is found to be

$$\phi(z) = \lambda r_{+(c)}\xi(z) \tag{4.14}$$

where

$$\lambda = \frac{\rho}{r^2_{+(c)}} \tag{4.15}$$

and

$$\xi(z) = \int_z^1 \frac{d\tilde{z}}{\sqrt{1 + b\lambda^2\tilde{z}^4}}. \tag{4.16}$$

In order to evaluate the above integration we shall perform a perturbative expansion of $b\lambda^2$ in the rhs of Eq.(4.16) and retain only the terms that are linear in b such that $b\lambda^2 = b\lambda_0^2 + O(b^2)$, where λ_0^2 is the value of λ^2 for $b = 0$. Now, for our particular choice of $\psi^{(i)}$ $(i = +, -)$ we have $\lambda_0^2 \approx 17.3$ [216]. Recalling that the existing values of b in the literature are $b = 0.1,\ 0.2,\ 0.3$ [224], we observe that $b\lambda_0^2 > 1$. Consequently, the binomial expansion of the denominator in Eq.(4.16) has to be done carefully. The integration appearing in Eq.(4.16) is done for two ranges of values of z, one for $z \leqslant \Lambda < 1$ while the other for $\Lambda \leqslant z \leqslant 1$, where Λ is such that $b\lambda_0^2 z^4|_{z=\Lambda} = 1$. At this stage, it is to be noted that $b\lambda_0^2 z^4 < 1$ for $z < \Lambda$, whereas, on the other hand $b\lambda_0^2 z^4 > 1$ for $z > \Lambda$.

Now, let us solve the integration of Eq.(4.16) for the two range of values of z mentioned above.

- For the range $z \leqslant \Lambda < 1$,

$$\begin{aligned}
\xi(z) = \xi_1(z) &= \int_z^{\Lambda} \frac{d\tilde{z}}{\sqrt{1+b\lambda_0^2\tilde{z}^4}} + \int_{\Lambda}^{1} \frac{d\tilde{z}}{\sqrt{1+b\lambda_0^2\tilde{z}^4}} \\
&\approx \int_z^{\Lambda} \left(1 - \frac{b\lambda_0^2\tilde{z}^4}{2}\right) + \frac{1}{\sqrt{b}\lambda_0}\int_{\Lambda}^{1}\left(\frac{1}{\tilde{z}^2} - \frac{1}{2b\lambda_0^2\tilde{z}^6}\right) \\
&= \left[\frac{9}{5}\Lambda - z + \frac{z^5}{10\Lambda^4} - \Lambda^2 + \frac{\Lambda^6}{10}\right]. \qquad (4.17)
\end{aligned}$$

- For the range $\Lambda \leqslant z \leqslant 1$,

$$\begin{aligned}
\xi(z) = \xi_2(z) &= \int_z^{1} \frac{d\tilde{z}}{\sqrt{1+b\lambda_0^2\tilde{z}^4}} \\
&\approx \frac{1}{\sqrt{b}\lambda_0}\int_z^{1}\left(\frac{1}{\tilde{z}^2} - \frac{1}{2b\lambda_0^2\tilde{z}^6}\right) \\
&= \frac{\Lambda^2}{z^5}\left[z^4(1-z) + \frac{\Lambda^4}{10}(z^5-1)\right]. \qquad (4.18)
\end{aligned}$$

From Eq.(4.18) it is evident that the boundary condition $\phi(1) = 0$ is indeed satisfied ($\xi_2(1) = 0$).

As a next step, let us express $\psi(z)$ near the boundary ($z \to 0$) as

$$\psi(z) = \frac{<\mathscr{O}>}{\sqrt{2}r_+^2} z^2 \mathscr{F}(z). \qquad (4.19)$$

Here $\mathscr{F}(z)$ is a trial function which satisfies the conditions $\mathscr{F}(0) = 1$ and $\mathscr{F}'(0) = 0$. This form of the trial function is compatible with the boundary behavior of the scalar field $\psi(z)$ (Eqn.(4.12)).

Using Eq. (4.19) we may write Eq. (4.9) as

$$\mathscr{F}''(z) - \frac{(5z^4 - 2z)}{z^2(1-z^3)}\mathscr{F}'(z) - \frac{4z^3}{z^2(1-z^3)}\mathscr{F}(z) + \lambda^2\frac{\xi^2(z)}{(1-z^3)^2}\mathscr{F}(z) = 0. \qquad (4.20)$$

This equation can be put in the Sturm-Liouville form as

$$[p(z)\mathscr{F}'(z)]' + q(z)\mathscr{F}(z) + \lambda^2 g(z)\mathscr{F}(z) = 0 \qquad (4.21)$$

with the following identifications:

$$\begin{aligned} p(z) &= z^2(1-z^3) \\ q(z) &= -4z^3 \\ g(z) &= \frac{z^2}{(1-z^3)}\xi^2(z) = \chi(z)\xi^2(z) \end{aligned} \tag{4.22}$$

where $\chi(z) = \dfrac{z^2}{(1-z^3)}$.

Using Eq.(4.22), we may write the eigenvalue λ^2 as

$$\begin{aligned} \lambda^2 &= \frac{\int_0^1 \{p(z)[\mathscr{F}'(z)]^2 - q(z)[\mathscr{F}(z)]^2\}\,dz}{\int_0^1 \{g(z)[\mathscr{F}(z)]^2\}\,dz} \\ &= \frac{\int_0^1 \{p(z)[\mathscr{F}'(z)]^2 - q(z)[\mathscr{F}(z)]^2\}\,dz}{\int_0^\Lambda \{\chi(z)\xi_1^2(z)[\mathscr{F}(z)]^2\}\,dz + \int_\Lambda^1 \{\chi(z)\xi_2^2(z)[\mathscr{F}(z)]^2\}\,dz}. \end{aligned} \tag{4.23}$$

Note that, we have chosen the trial function in the form

$$\mathscr{F}(z) = 1 - \alpha z^2 \tag{4.24}$$

which satisfies the conditions $\mathscr{F}(0) = 1$ and $\mathscr{F}'(0) = 0$. In this regard we must point out one important difference between our analytic approach and the numerical method existing in the literature[224]. While solving Eqs. (4.9) and (4.13) numerically the functions $\mathscr{F}(z)$ and $\xi(z)$ (Eq.(4.16)) do not appear in the analysis, instead they are solved directly. On the contrary, in the analytic method certain approximations are needed to solve Eqs. (4.9) and (4.13) which however demonstrates the limitations of the analytic method adopted here.

The critical temperature for condensation (T_c) in terms of the charge density (ρ) can be obtained as

$$T_c = \frac{3r_{+(c)}}{4\pi} = \gamma\sqrt{\rho}. \tag{4.25}$$

In Eq.(4.25) $\gamma = \dfrac{3}{4\pi\sqrt{\lambda}}$ is the coefficient of T_c and we have used Eqs. (4.3) and (4.15) to arrive at this relation.

Let us now determine the eigenvalue (λ) of Eq.(4.23) for different values of the parameter b which will enable us to demonstrate the effect of higher derivative gauge correction on the critical temperature through Eq.(4.25). As an example, for $b = 0.1$, we obtain

$$\lambda^2 = 300.769 + \frac{2.27395\alpha - 5.19713}{0.0206043 + (0.00265985\alpha - 0.0119935)\alpha} \tag{4.26}$$

which has a minima for $\alpha \approx 0.653219$. Therefore from eq.(4.23) we obtain

$$\lambda^2 \approx 33.8298. \tag{4.27}$$

The value of λ^2 obtained from the perturbative calculation justifies our approximation for computing the integral in Eq.(4.16) upto order b and neglecting terms of order b^2 and higher since the term of order b^2 can be estimated to be smaller than the term of order b.

Substituting Eq.(4.27) in Eq.(4.25) we obtain,

$$T_c \approx 0.099\sqrt{\rho}. \tag{4.28}$$

The value thus obtained analytically is indeed in very good agreement with the numerical result: $T_c = 0.10072\sqrt{\rho}$ [224]. Similarly, for the other values of the Born-Infeld parameter (b), we obtain the corresponding perturbative values for the coefficients of T_c which are presented in the Table 4.1 below.

Values of b	$\gamma_{numerical}$	γ_{SL}
0.1	0.10072	0.099
0.2	0.08566	0.093
0.3	0.07292	0.089

Table 4.1: A comparison between analytic and numerical values for the coefficient (γ) of T_c corresponding to different values of b

Before concluding this section, we would like to emphasize the subtlety of the analytic method adapted here. We employ a perturbative technique to compute the integral in Eq.(4.16) upto order b. This approximation is valid since we have investigated the effect of the higher derivative corrections upto the leading order in the nonlinear parameter (b). However, due to the nature of the integrand of Eq.(4.16), we had to be careful in separating the integral in two regions in order to perform a binomial expansion of the integrand.

4.4 Order parameter for condensation

In the present work, the order parameter ($\langle\mathscr{O}\rangle$) for the s-wave condensate in the boundary field theory has been chosen as the subleading coefficient of the near boundary behavior of the scalar field $\psi(z)$ (cf. Eq.(4.12)). In order to calculate the order parameter we need to consider the behavior of the gauge field (ϕ) near the critical temperature T_c. Substituting Eq.(4.19) into Eq.(4.10) we can find

$$\phi''(z) + \frac{2bz^3}{r_+^2}\phi'^3(z) = \frac{\mathscr{F}^2(z)z^2\langle\mathscr{O}\rangle^2}{r_+^4(1-z^3)}\left(1 - \frac{3bz^4\phi'^2(z)}{2r_+^2}\right)\phi(z) + \mathbf{O}(b^2) \tag{4.29}$$

It is to be noted that in the subsequent analysis only terms upto linear order in b have been taken into account.

As a next step, we expand $\phi(z)$ perturbatively in the small parameter $\langle\mathscr{O}\rangle^2/r_+^4$ as follows :

$$\frac{\phi(z)}{r_+} = \frac{\phi_0(z)}{r_+} + \frac{\langle\mathscr{O}\rangle^2}{r_+^4}\chi(z) + \text{higher order terms.} \tag{4.30}$$

where ϕ_0 is the solution of Eq.(4.13). Here $\chi(z)$ is some arbitrary function which satisfies the boundary condition

$$\chi(1) = \chi'(1) = 0. \tag{4.31}$$

Substituting Eq.(4.30) and Eq.(4.14) we may write Eq.(4.29) in terms of $\chi(z)$ as

$$\chi''(z) + 6b\lambda^2 z^3\xi'^2(z)\chi'(z) = \lambda\frac{\mathscr{F}^2(z)z^2\xi(z)}{(1-z^3)}\left(1 - \frac{3b\lambda^2 z^4\xi'^2(z)}{2}\right) \tag{4.32}$$

Multiplying both sides of Eq.(4.32) by $e^{3b\lambda^2 z^4\xi'^2(z)/2}$ we obtain

$$\frac{d}{dz}\left(e^{3b\lambda^2 z^4\xi'^2(z)/2}\chi'(z)\right) = e^{3b\lambda^2 z^4\xi'^2(z)/2}\lambda\frac{\mathscr{F}^2(z)z^2\xi(z)}{(1-z^3)}\left(1 - \frac{3b\lambda^2 z^4\xi'^2(z)}{2}\right) \tag{4.33}$$

The **l.h.s** of Eq.(4.33) may be written as,

$$\frac{d}{dz}\left(e^{3b\lambda^2 z^4\xi'^2(z)/2}\chi'(z)\right) = e^{3b\lambda^2 z^4\xi'^2(z)/2}\left[\chi''(z) + 6b\lambda^2 z^3\xi'^2(z)\chi'(z) + 3b\lambda^2 z^4\xi'(z)\xi''(z)\chi'(z)\right]. \tag{4.34}$$

The last term in the r.h.s of Eq.(4.34) can be rewritten as,

$$\begin{aligned} 3b\lambda^2 z^4\xi'(z)\xi''(z)\chi'(z) &= 3b\lambda^2 z^4\xi'(z)\left(\frac{\phi_0''(z)}{\lambda r_+}\right)\chi'(z) \\ &= \frac{-6b^2\lambda z^7\xi'(z)}{r_+^3}\phi_0'^3(z)\chi'(z) \\ &= -6b^2\lambda^4 z^7\xi'^4(z)\chi'(z) \\ &\approx 0. \end{aligned} \tag{4.35}$$

where we have used Eq.(4.13).

Therefore Eqn.(4.34) becomes

$$\frac{d}{dz}\left(e^{3b\lambda^2 z^4\xi'^2(z)/2}\chi'(z)\right) = e^{3b\lambda^2 z^4\xi'^2(z)/2}\left[\chi''(z) + 6b\lambda^2 z^3\xi'^2(z)\chi'(z)\right]. \tag{4.36}$$

The **r.h.s** of Eq.(4.33) may be written as,

$$\begin{aligned} e^{3b\lambda^2 z^4\xi'^2(z)/2}\frac{\lambda\mathscr{F}^2(z)z^2\xi(z)}{(1-z^3)}\left(1 - \frac{3b\lambda^2 z^4\xi'^2(z)}{2}\right) &\approx \left(1 + \frac{3b\lambda^2 z^4\xi'^2(z)}{2}\right)\frac{\lambda\mathscr{F}^2(z)z^2\xi(z)}{(1-z^3)} \\ &\quad \left(1 - \frac{3b\lambda^2 z^4\xi'^2(z)}{2}\right) \\ &\approx \frac{\lambda\mathscr{F}^2(z)z^2\xi(z)}{(1-z^3)}. \end{aligned} \tag{4.37}$$

Combining Eq.(4.36) and Eq.(4.37) we obtain

$$\frac{d}{dz}\left(e^{3b\lambda^2 z^4 \xi'^2(z)/2}\chi'(z)\right) = \lambda\frac{\mathscr{F}^2(z)z^2\xi(z)}{(1-z^3)}. \tag{4.38}$$

Using the boundary condition Eq.(4.31) and integrating Eq.(4.38) in the interval [0, 1] we finally obtain

$$\chi'(0) = -\lambda\left(\mathscr{A}_1 + \mathscr{A}_2\right) \tag{4.39}$$

where,

$$\begin{aligned}
\mathscr{A}_1 = &\int_0^\Lambda \frac{\mathscr{F}^2(z)z^2\xi_1(z)}{(1-z^3)} \qquad\qquad , for\ \ 0 \leq z < \Lambda \\
= &\frac{1}{12600\Lambda^4}\{-70\sqrt{3}\pi(-1-10\Lambda^4+\alpha(-2+\Lambda^4(10\alpha+18(2+\alpha)\Lambda-10(2+\alpha)\Lambda^2+ \\
&(2+\alpha)\Lambda^6))) + \Lambda(126\Lambda(-5+98\Lambda^3)+30\alpha(84+\Lambda^3(21+2\Lambda^3(244+21\Lambda(-10 \\
&+\Lambda^4)))) - 35\alpha^2\Lambda^2(12+\Lambda^3(474+\Lambda(-360+\Lambda^2(94+9\Lambda(-10+4\Lambda+\Lambda^4)))))) \\
&-420\ \log(1-\Lambda)+210\ log(1+\Lambda+\Lambda^2)+210(2\sqrt{3}(-1-10\Lambda^4+\alpha(-2+\Lambda^4(10\alpha \\
&+18(2+\alpha)\Lambda-10(2+\alpha)\Lambda^2+(2+\alpha)\Lambda^6)))\ tan^{-1}\left(\frac{1+2\Lambda}{\sqrt{3}}\right)-(-2\alpha-10(1+\alpha^2) \\
&\Lambda^4+18(\alpha-2)\alpha\Lambda^5-10(\alpha-2)\alpha\Lambda^6+(\alpha-2)\alpha\Lambda^{10})(2\ log(1-\Lambda)-\ \log(1+\Lambda+\Lambda^2)) \\
&-2(\alpha^2+20\alpha\Lambda^4+\Lambda^5(18-10\Lambda+\Lambda^5))\ \log(1-\Lambda^3))\}
\end{aligned} \tag{4.40}$$

and

$$\begin{aligned}
\mathscr{A}_2 = &\int_\Lambda^1 \frac{\mathscr{F}^2(z)z^2\xi_2(z)}{(1-z^3)} \qquad\qquad , for\ \ \Lambda < z \leq 1 \\
= &\frac{\Lambda^2}{360}\{4\sqrt{3}\pi(-10(1+\alpha(4+\alpha))+(-1+2\alpha(1+\alpha))\Lambda^4)+12\sqrt{3}(10+10\alpha(4+\alpha)+\Lambda^4 \\
&-2\alpha(1+\alpha)\Lambda^4)\ tan^{-1}\left(\frac{1+2\Lambda}{\sqrt{3}}\right)+3(-12\alpha(-10+\Lambda(20-10\Lambda+\Lambda^5+\Lambda^3(-1+\ \log 3))) \\
&-6(\Lambda^2+\Lambda^4(-1+\ \log 3))+\alpha^2(110+\Lambda(-120+\Lambda^2(40+3\Lambda(-15+4\Lambda+\Lambda^4)))-60\ \log 3) \\
&+60\ \log 3-24\alpha\Lambda^4\log\Lambda+6(-10+\Lambda^4+2\alpha(5\alpha+\Lambda^4))\ \log(1+\Lambda+\Lambda^2))\}
\end{aligned} \tag{4.41}$$

where $\xi_1(z)$ and $\xi_2(z)$ were identified earlier (Eqn.(4.17) and Eqn.(4.18)).

Now from Eq. (4.11) and Eq.(4.30) we may write

$$\begin{aligned}
\frac{\mu}{r_+} - \frac{\rho}{r_+^2}z &= \frac{\phi_0(z)}{r_+} + \frac{\langle\mathscr{O}\rangle^2}{r_+^4}\chi(z) \\
&= \lambda\,\xi(z) + \frac{\langle\mathscr{O}\rangle^2}{r_+^4}\left\{\chi(0) + z\,\chi'(0) + \frac{z^2}{2!}\chi''(0) + ...\right\}
\end{aligned} \tag{4.42}$$

It is to be noted that, while writing the r.h.s of Eq.(4.42) we have made a Taylor expansion of $\chi(z)$ around $z = 0$.

Comparing the coefficients of z from both sides of Eq.(4.42), we obtain

$$\frac{\rho}{r_+^2} = \lambda - \frac{\langle \mathscr{O} \rangle^2}{r_+^4} \chi'(0). \tag{4.43}$$

Substituting Eq.(4.39) we may write Eq.(4.43) in the following form:

$$\frac{\rho}{r_+^2} = \lambda \left\{ 1 + \frac{\langle \mathscr{O} \rangle^2}{r_+^4} (\mathscr{A}_1 + \mathscr{A}_2) \right\}. \tag{4.44}$$

Substituting $\lambda = \rho / r_{+(c)}^2$ (cf. Eq.(4.14)) into Eq.(4.44) we finally obtain the expression for the order parameter $\langle \mathscr{O} \rangle$ near the critical temperature (T_c) as,

$$\langle \mathscr{O} \rangle = \beta \, T_c^2 \sqrt{1 - \frac{T}{T_c}} \tag{4.45}$$

where the coefficient β is given by,

$$\beta = \frac{16\sqrt{2}\, \pi^2}{9\sqrt{(\mathscr{A}_1 + \mathscr{A}_2)}}. \tag{4.46}$$

In the following table (Table 2) we have provided both analytic as well as numerical[224] values for the coefficient β corresponding to different values of the Born-Infeld parameter (b).

Table 4.2: Values of β (Eq.(4.46)) for different values of b

Values of b	Values of α	$(\mathscr{A}_1 + \mathscr{A}_2)$	β_{SL}	$\beta_{numerical}$
0.1	0.653219	0.0442811	117.919	207.360
0.2	0.656050	0.0388491	125.893	302.760
0.3	0.660111	0.0352282	132.205	432.640

Here (from Table 4.2) one can note that both the values that are obtained through different approaches are in the same order. The difference that is caused is mainly due to the perturbative technique itself where we have dropped higher order terms in the coupling (b). Similar features have also been found earlier[221]. However, the trend is unique, i.e., β increases as we increase the value of coupling b (see also Fig.(4.1)). Indeed, it would be interesting to carry out the analysis taking into account higher order terms in the coupling b which is expected to reduce the disparity between the analytic and numerical results.

4.5 Conclusive remarks

In this chapter we have discussed the principle properties of a holographic model of superconductor based on fundamental principles of AdS/CFT duality. Among

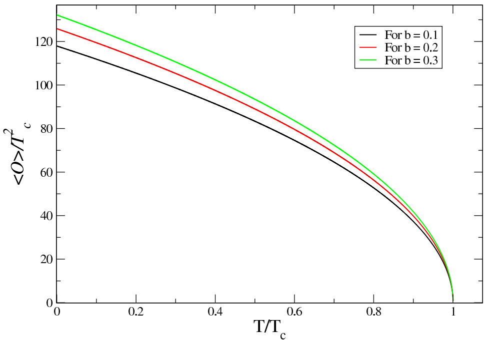

Figure 4.1: *Plot of* $\langle \mathcal{O} \rangle / T_c^2$ *with* T/T_c *for different values of* b.

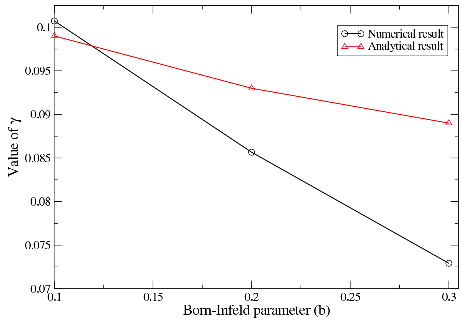

Figure 4.2: *Plot of the coefficient of* T_c (γ) *with the BI parameter* (b).

several models of holographic superconductor in the AdS black hole background, we have taken into account a model in which nonlinear Born-Infeld Lagrangian is included in the matter action. The main purpose for considering the BI theory is that it corresponds to the higher derivative corrections of the gauge fields in the usual Abelian theory that effectively describes the *low energy* behavior of the string theory. In this sense it may be considered as the generalized version of the Abelian model. These corrections must have nontrivial influences on the physical properties of the system.

The aim of the present analysis is to study the effects of these higher derivative corrections on the holographic s-wave condensate analytically. In this regard we have been able to extend the so called Sturm-Liouville (SL) method for this nonlinear model. This method was first introduced in [216] in the context of usual Maxwell theory. From our analysis it is indeed evident that extending such a method for the nonlinear model creates difficulties in the analysis. However, we have been able to construct an analytic technique based on this SL method in order to analyse the properties of this holographic superconductor subjected to a nontrivial boundary condition. On top of it, our approach reveals the fact that the solutions of the field equations are highly nontrivial and are not even exactly solvable. The analytic method presented here provides a smooth platform to deal with this difficulty.

The novelty of the present analysis is that, we have analytically studied the effects of the BI coupling parameter b on the critical temperature and the condensation operator near the critical point. It is observed that the above physical quantities are indeed affected due to the higher derivative corrections. The results thus obtained from our calculations can be summarized qualitatively as follow:

• The critical temperature (T_c) increases as we decrease the value of b indicating the onset of a harder condensation (Table 4.1).

• The value of the order parameter increases with the increase of b (Table 4.2).

The point that must be stressed at this stage of discussion is that the analytic approach is always more preferable than the numerical approach. This is due to the fact that the numerical results become less reliable when the temperature T approaches to zero[196, 216]. In this temperature limit the numerical solutions to the nonlinear field equations becomes very much difficult and therefore the determination of the nature of the condensate becomes practically very arduous unless analytic methods are taken into account. Therefore, analytic method is always more reliable while performing computations as $T \to 0$.

From Fig. 4.2 one may note that the results that are obtained using the SL method are not exactly identical to those obtained by numerical techniques. The deviation of the analytic values from those of the numerical one (Tables 4.1 and 4.2) is not unusual[221], considering the difference in the two approaches (analytic and numerical). Contrary to the numerical approach, in the analytic method we have taken into account only the leading order terms in the coupling b. Certainly, there is a great amount of approximation involved which is absent in the numerical technique. The difference between the two approaches in fact motivates us to enquire into a more general analytic approach in which the above disparity may be reduced and the agreement eventually becomes more close.

In passing, we would like to mention that the numerical results obtained in the existing literature have always been substantiated by analytic results. However, one may confirm the validity of the analytic results obtained by the Sturm-Liouville (SL) method (without referring to the numerical results) by comparing them with the results obtained from an alternative analytic technique which is known as the matching method[213].

In order to continue our study of holographic superconductors, in the next chapter, we shall consider a Gauss-Bonnet holographic superconductor in the presence of several Born-Infeld-like corrections. We shall investigate the effects of non-linearity on this model and shall make a comparison between these two types of corrections.

Chapter 5

Gauge and Gravity Corrections to Holographic Superconductors: A Comparative Survey

5.1 Overview and motivations

The systematic exposition of holographic superconductors in the presence of higher derivative corrections to the Maxwell electrodynamics presented in the previous Chapter 4 demonstrates the power of the AdS/CFT duality in describing various aspects of strongly coupled condensed matter systems which is otherwise difficult to explain in the usual framework of perturbation theory. We analyzed properties of holographic superconductors in Chapter 4 by considering only non-linear corrections to the usual gauge fields in the Abelian-Higgs action (see Eq. (4.4), Chapter 4). However, there remains ample scopes for studying these holographic models in presence of other corrections to the gauge and/or gravity sector of the action Eq. (1.6). A extensive amount of research has been performed in this direction[217]-[223],[225]-[227],[229, 232],[234]-[240]. These studies reveal that these non-linear corrections modify the physical quantities of interest of a HS. Along with this, holographic superconductors in the presence of higher curvature corrections (e.g., Gauss-Bonnet correction) seem to posses several unique features such as, *(i)* the observed constancy of the ratio of the frequency gap of the real part of the conductivity to the critical temperature of the superconductor[206] breaks down for Gauss-Bonnet superconductors[213, 215, 218, 220], *(ii)* in certain generalized cases of different values of the Gauss-Bonnet correction changes the order of the phase transition[218], *(iii)* the ratio of the shear viscosity to the entropy density ($\eta/s \geqslant 1/4\pi$) in CFT dual to the Einstein-Gauss-Bonnet gravity changes significantly with the Gauss-Bonnet coupling[186].

Here we must mention that none of these studies takes into account both the corrections simultaneously. Thus it remains an open issue to make a comparative study between these two corrections regarding their effects of the holographic condensates. In the present chapter, based on Ref.[240], we aim to address this issue by incorporating both Gauss-Bonnet higher curvature correction to the gravity sector and higher derivative Born-Infeld *like* corrections in the Abelian-Higgs sector.[1]

Higher curvature gravity theories have earned repeated attentions in recent past with the advent of string theory. Certain aspects of string theory are well described by associating gravity theories with it. The consistent description of string theory requires the addition of higher curvature terms in the effective action. In fact the effect of string theory on gravity may be studied by considering this low-energy effective action which describes classical gravity[4]. It has been observed that this effective action must contain higher curvature terms and are needed to be ghost free[16]. The Lovelock action is found to be consistent with these criteria[5]. On the other hand, the importance of higher derivative corrections to the Maxwell electrodynamics has been highlighted repeatedly for the past several decades. The primary motivation for introducing this latter correction was to remove divergences in the self-energy of point-like charged particles[20]. However, they have earned renewed attentions since these theories naturally arise string theory[35].

Besides the conventional Born-Infeld non-linear electrodynamics (BINE)[20], two new types of NEDs have been proposed recently, namely, the logarithmic non-linear electrodynamics (LNE) and the exponential non-linear electrodynamics (ENE), in the context of static charged asymptotic black holes[37, 38, 39]. In fact, the matter actions with LNE and ENE yield the higher derivative corrections to the usual Maxwell action. On the other hand these NEDs possess many unique properties which are quite different from the Maxwell electrodynamics. For example, while solutions with LNE completely remove divergences in the electric fields at $r = 0$, these divergences still remain in the solutions with ENE. But these divergences are much weaker than the usual Maxwell case[38, 39]. Also, compared with Maxwell theory, solutions with LNE and ENE have different temperatures and electric potentials[37, 38, 39]. Another novel property of these non-linear theories is that, their asymptotic black hole solutions are the same as that of a Reissner-Nordström black hole[39]. On top of that, these types of non-linear theories retain some interesting properties (alike BINE) such as, absence of shock waves, birefringence etc.[38, 39]. One further advantage of studying ENE and LNE over the Maxwell theory is that they provide an enriched platform to investigate generalized versions of NEDs in a systematic manner so as to reveal some general features of the effects of higher derivative corrections to the gauge fields in the theory concerned.

Apart from these, the effects of external magnetic fields on the holographic superconductors with or without these nontrivial non-linear corrections have been studied. These studies reveal several interesting properties of these superconductors which resemble certain properties of conventional superconductors, such as the Meissner effect, vortex and droplet solutions etc.[245]-[258].

It must be emphasized that holographic superconductors with several NEDs (BINE, ENE, LNE) has been studied in Ref.[236] in the planar Schwarzschild-AdS black hole background without taking into account higher curvature corrections to

[1] In this regard we should mention the work of Ref.[239] which appeared almost simultaneously with Ref.[240].

the Einstein gravity. Also, in Ref.[239] a holographic model with BINE in the Gauss-Bonnet gravity has been considered. Surprisingly most of the computations have been performed using numerical methods. Therefore, it would be nice to perform analytic analysis of holographic superconductor models with non-linear electrodynamic fields (ENE and LNE) in the Gauss-Bonnet black hole background.

Considering all the above mentioned facts, in this chapter we have made an extensive analytic investigation of the holographic model of superconductors mentioned at the end of the previous paragraph in the presence as well as absence of an external magnetic field. Our aim is to address the following issues: *(i)* investigate the effects of higher curvature as well as higher derivative corrections on the properties of holographic superconductors, *(ii)* explore the effects of an external magnetic field on the holographic condensates, and determine how the non-linear corrections modify the critical value of the magnetic field, *(iii)* make a comparison between the Maxwell theory and the non-linear electrodynamic theories regarding their effects on the formation of the scalar condensates, *(iv)* compare the non-linear electrodynamic theories in an attempt to see which one has stronger effects on the formation of the scalar condensates.

The present chapter is organized as follows. In Section 5.2, we shall present the basic setup for s-wave holographic superconductor with two different non-linear electrodynamics (ENE and LNE) in the planar $(4+1)$-dimensional Gauss-Bonnet AdS black hole background. In Section 5.3, we shall calculate various properties of the s-wave holographic superconductor with exponential electrodynamics (ENE) which include the critical temperatures for condensation and the expectation values of the condensation operator in the absence of external magnetic field. In Section 5.4, we'll discuss the effects of an external static magnetic field on this holographic superconductor and calculate the critical magnetic field for condensation. Finally, we draw our conclusions in Section 5.5.

5.2 Basic set up

In this Chapter we consider both gravity and gauge corrections to the usual action for holographic superconductors (cf. Eq. (1.6) of Chapter 1). Here, as gravity correction we consider higher curvature Gauss-Bonnet (GB) correction to the usual Einstein-Hilbert action. On the other hand, exponential and logarithmic corrections to the usual Maxwell gauge fields are taken into account as gauge corrections which are important variants of the well known Born-Infeld corrections to the Abelian gauge fields.

The general form of the action containing the Gauss-Bonnet correction can be obtained as the truncation of the Lovelock action which is given by[5]-[19],

$$\mathscr{S}_{grav} = \frac{1}{16\pi G_d}\int d^dx\sqrt{-g}\sum_{i=0}^{[d/2]}\alpha_i\mathscr{L}_i \tag{5.1}$$

where, α_i is an arbitrary constant, $\mathscr{L}_i$ is the *Euler density* of a $2i$ dimensional manifold and G_d is the Gravitational constant in d-dimensions. In our subsequent analysis we shall consider the coordinate system where $G_d = \hbar = k_B = c = 1$. Now, since the GB term contributes non-trivially to the equations of motion only when space-time dimension is greater than four ($D > (3+1)$), we are mainly concerned with the $(4+1)$-dimensional Einstein-Gauss-Bonnet gravity in anti-de Sitter (AdS) space. Thus, the effective action Eq. (5.1) can be written as,

$$\begin{aligned} \mathscr{S}_{grav} &= \frac{1}{16\pi}\int d^5x\sqrt{-g}\Big(\alpha_0\mathscr{L}_0 + \alpha_1\mathscr{L}_1 + \alpha_2\mathscr{L}_2\Big) \\ &= \frac{1}{16\pi}\int d^5x\sqrt{-g}\Big(-2\Lambda + \mathscr{R} + \alpha\mathscr{L}_2\Big) \end{aligned} \tag{5.2}$$

where Λ is the cosmological constant given by $-6/l^2$, l being the AdS length, $\alpha_2 \equiv \alpha$ is the Gauss-Bonnet coefficient, $\mathscr{L}_1 = \mathscr{R}$ is the usual Einstein-Hilbert Lagrangian and $\mathscr{L}_2 = \big(R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + \mathscr{R}^2\big)$ is the Gauss-Bonnet Lagrangian.

The Ricci flat solution for the action Eq. (5.2) is given by[89, 213]

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(dx^2 + dy^2 + dz^2) \tag{5.3}$$

where the metric function is[2][89, 213],

$$f(r) = \frac{r^2}{2a}\left(1 - \sqrt{1 - 4a\left(1 - \frac{M}{r^4}\right)}\right). \tag{5.4}$$

In Eq. (5.4), M is the mass of the black hole which may be expressed in terms of the horizon radius (r_+) as, $r_+ = M^{1/4}$[89, 213, 215, 229]; the parameter a is related to the coefficient α as, $a = 2\alpha$. It is to be noted that, in order to avoid naked singularity we must have $a \leq 1/4$[213, 229], whereas, considering the causality of dual field theory on the boundary the lower bound of a is found to be $a \geq -7/36$[218, 219]. Also, in the asymptotic infinity ($r \to \infty$) we may write the metric function Eq. (5.4) as,

$$f(r) \sim \frac{r^2}{2a}\Big(1 - \sqrt{1 - 4a}\Big). \tag{5.5}$$

Thus, the effective AdS radius can be defined as[213],

$$L_{eff}^2 = \frac{2a}{1 - \sqrt{1 - 4a}}. \tag{5.6}$$

Note that, in the limit $a \to 1/4$, $L_{eff}^2 = 0.5$. This limit is known as the Chern-Simons limit[213, 229].

The Hawking temperature of the black hole may be obtained by analytic continuation of the metric at the horizon (r_+) and is given by[89, 213, 215, 229],

$$T = \frac{r_+}{\pi}. \tag{5.7}$$

[2]Without loss of generality, we can choose $l = 1$, which follows from the scaling properties of the equation of motion.

In this chapter we will be studying the s-wave holographic superconductor in the framework of various non-linear electrodynamics in the $(4+1)$-dimensional planar Gauss-Bonnet AdS (GBAdS) black hole background. For this purpose, we consider a matter Lagrangian which consists of a charged $U(1)$ gauge field, A_μ, and a charged massive complex scalar field, ψ. Thus, following Refs. [191]-[197], the matter action for the theory may be written as,

$$\mathscr{S}_{matter} = \int d^5x\sqrt{-g}\Big(L(F) - |\nabla_\mu\psi - iA_\mu\psi|^2 - m^2\psi^2\Big) \tag{5.8}$$

where m is the mass of the scalar field. Moreover, we carry out all the calculations in the probe limit which was discussed in Chapter 4. The term $L(F)$ in Eq. (5.8) corresponds to the Lagrangian for the non-linear electrodynamic field. In different non-linear theories the Lagrangian $L(F)$ can take the following forms[37, 38, 39]:

$$L(F) = \begin{cases} \dfrac{1}{4b}\Big(e^{-bF^{\mu\nu}F_{\mu\nu}} - 1\Big), & \text{for ENE} \\[2ex] \dfrac{-2}{b}\, ln\Big(1 + \frac{1}{8}bF^{\mu\nu}F_{\mu\nu}\Big), & \text{for LNE} \end{cases} \tag{5.9}$$

Note that, in the limit $b \to 0$ we recover the usual Maxwell Lagrangian: $L(F)|_{b\to 0} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$.

In order to solve the equations of motion resulting from the variation of the action Eq. (5.8) w.r.to the gauge and scalar fields we shall choose the following ansatz for the two fields concerned[213, 229]:

$$A_\mu = \Big(\phi(r), 0, 0, 0, 0\Big), \tag{5.10a}$$

$$\psi = \psi(r). \tag{5.10b}$$

It is to be noted that, the above choices of the fields are justified since it is seen that under the transformations $A_\mu \to A_\mu + \partial_\mu\theta$ and $\psi \to \psi e^{i\theta}$ the above equations of motion remain invariant. This demands that the phase of ψ remains constant and we may take ψ to be real without any loss of generality.

With the change of coordinates $z = \frac{r_+}{r}$, where the horizon $(r = r_+)$ is at $z = 1$ and the boundary $(r \to \infty)$ is at $z = 0$, the equations of motion for the the scalar field $(\psi(z))$ and the $U(1)$ gauge field (A_μ) corresponding to ENE and LNE can be obtained as,

$$\psi''(z) + \left(\frac{f'(z)}{f(z)} - \frac{1}{z}\right)\psi'(z) + \frac{\phi^2(z)\psi(z)r_+^2}{z^4 f^2(z)} - \frac{m^2\psi(z)r_+^2}{z^4 f(z)} = 0, \tag{5.11}$$

$$\left(1 + \frac{4bz^4\phi'^2(z)}{r_+^2}\right)\phi''(z) - \frac{1}{z}\phi'(z) + \frac{8bz^3\phi'^3(z)}{r_+^2} - \frac{2\psi^2(z)\phi(z)}{r_+^{-2}f(z)z^4}e^{-2bz^4\phi'^2(z)/r_+^2} = 0, \tag{5.12}$$

$$\left(1+\frac{bz^4\phi'^2(z)}{4r_+^2}\right)\phi''(z)-\frac{\phi'(z)}{z}+\frac{5bz^3\phi'^3(z)}{4r_+^2}-\frac{2\psi^2(z)\phi(z)}{r_+^{-2}f(z)z^4}\left(1-\frac{bz^4\phi'^2(z)}{4r_+^2}\right)^2=0, \tag{5.13}$$

respectively. In order to solve the above set of equations we consider the following boundary conditions:

(i) At the horizon ($z=1$) one must have, for $m^2=-3$,[3]

$$\phi(1)=0, \quad \psi^{'}(1)=\frac{3}{4}\psi(1) \tag{5.14}$$

(ii) In the asymptotic AdS region ($z\to 0$) the solutions for the scalar potential and the scalar field may be expressed as,

$$\phi(z)=\mu-\frac{\rho}{r_+^2}z^2, \tag{5.15a}$$

$$\psi(z)=D_-z^{\lambda_-}+D_+z^{\lambda_+} \tag{5.15b}$$

where $\lambda_\pm=2\pm\sqrt{4-3L_{eff}^2}$ is the conformal dimension of the condensation operator $\mathscr{O}_i$ ($i=1,2$) in the boundary field theory, μ and ρ are identified as the chemical potential and the charge density of the dual field theory, respectively. It is interesting to note that, since we have considered $m^2<0$ in our analysis, we are left with the two different condensation operators of different dimensionality corresponding to the choice of quantization of the scalar field ψ in the bulk. We choose $D_-=0$. Then, according to the AdS/CFT correspondence $D_+\equiv\langle\mathscr{O}_2\rangle$, the expectation value of the condensation operator in the dual field theory.

5.3 s-wave condensation without magnetic field

In this section we aim to derive the critical temperature for condensation (T_c) analytically for the Gauss-Bonnet holographic s-wave condensate with two types of non-linear electrodynamics mentioned in the previous section. In addition, we will also calculate the normalized condensation operator and the critical exponent associated with the condensation values in the presence of these non-linear theories in the background of $(4+1)$-dimensional Gauss-Bonnet AdS black hole. In this way we will be able to demonstrate the effects of the Gauss-Bonnet coupling coefficient (a) as well as non-linear parameter (b) on these condensates.

In order to carry out our analysis we adopt a well known analytic technique which is known as the matching method[213]. In this method, we first determine the leading order solutions of the equations of motion Eqs. (5.11), (5.12) and (5.13)

[3]For the rest of the analysis of our paper we choose $m^2=-3$. This ensures that we are above the Breitenlohner-Freedman bound[242].

near the horizon ($1 \geq z > z_m$) and at the asymptotic infinity ($z_m > z \geq 0$) and then match these solutions smoothly at the intermediate point, z_m. At this stage of discussion it is worthy of mentioning that, we pursue our analytic investigation in the same spirit as in Refs.[213], [234] and match the leading order solutions near the horizon and the boundary at the intermediate point $z_m = 0.5$. It may be stressed that the qualitative features of the analytical approximation does not change for other values of z_m ($0 < z_m \leq 1$) and differences in the numerical values are not too large[213]. Therefore throughout our analysis we shall choose $z_m = 0.5$ while obtaining numerical values and plotting various quantities. In fact, as mentioned in Ref.[222], this choice of the matching point roughly corresponds to the geometrical mean of the horizon radius and the AdS scale. Interestingly, with this choice of z_m our results are fairly consistent with Ref.[213] for $b = 0$. It is also to be noted that, the matching method helps us to determine the values of the critical temperature as well as of condensation operator only approximately, in the leading order of the non-linear parameter, b. Moreover, this method provides us a much better understanding of the effects the Gauss-Bonnet coefficient (a) as far as analytic computation is concerned[213].

In this section we present the detail analysis for the holographic s-wave superconductor with exponential electrodynamics only. Since, the analysis for the holographic superconductor with logarithmic electrodynamics closely resemblances to that of the previous one, we shall only write down the corresponding expressions for this model (below Eq. (5.32)).

Let us first consider the solutions of the gauge field, $\phi(z)$, and the scalar field, $\psi(z)$, near the horizon ($z = 1$). We Taylor expand both $\phi(z)$ and $\psi(z)$ near the horizon as[213],

$$\phi(z) = \phi(1) - \phi^{'}(1)(1-z) + \frac{1}{2}\phi^{''}(1)(1-z)^2 + \cdots \tag{5.16}$$

$$\psi(z) = \psi(1) - \psi^{'}(1)(1-z) + \frac{1}{2}\psi^{''}(1)(1-z)^2 + \cdots \tag{5.17}$$

It is to be noted that, in Eq. (5.16) and Eq. (5.17), we have considered $\phi'(1) < 0$ and $\psi(1) > 0$ in order to make $\phi(z)$ and $\psi(z)$ positive. This can be done without any loss of generality.

Near the horizon, $z = 1$, we may write from Eq. (5.11)

$$\psi^{''}(1) = \left[\frac{1}{z}\psi^{'}(z)\right]_{z=1} - \left[\frac{f^{'}(z)\psi^{'}(z)}{f(z)}\right]_{z=1} - \left[\frac{\phi^2(z)\psi(z)r_+^2}{z^4 f^2(z)}\right]_{z=1} - \left[\frac{3\psi(z){r_+}^2}{z^4 f(z)}\right]_{z=1}. \tag{5.18}$$

Using the L'Hôpital's rule and the values $f^{'}(1) = -4r_+^2$, $f^{''}(1) = 4{r_+}^2 + 32a{r_+}^2$, we may express Eq. (5.18) in the following form:

$$\psi^{''}(1) = -\frac{5}{4}\psi^{'}(1) + 8a\psi^{'}(1) - \frac{\phi^{'2}(1)\psi(1)}{16{r_+}^2}. \tag{5.19}$$

Finally, using the boundary condition Eq. (5.14), the Taylor expansion Eq. (5.17) can be rewritten as,

$$\psi(z) = \frac{1}{4}\psi(1) + \frac{3}{4}\psi(1)z + (1-z)^2\left[-\frac{15}{64} + \frac{3a}{2} - \frac{\phi'^2(1)}{64{r_+}^2}\right]\psi(1). \tag{5.20}$$

Near the horizon, $z = 1$, from Eq. (5.12) we may write

$$\phi''(1) = \frac{1}{\left(1 + \frac{4b}{r_+^2}\phi'^2(1)\right)}\left[\phi'(1) - \frac{8b}{{r_+}^2}\phi'^3(1) - \frac{\psi^2(1)\phi'(1)}{2}e^{-2b\phi'^2(z)/r_+^2}\right]. \tag{5.21}$$

In obtaining Eq. (5.21) we have considered that the metric function, $f(z)$, can also be Taylor expanded as in Eq. (5.16), Eq. (5.17).

Substituting Eq. (5.21) in Eq. (5.16) and using Eq. (5.14) we finally obtain,

$$\phi(z) = -\phi'(1)(1-z) + \frac{1}{2}(1-z)^2\left[1 - \frac{8b}{{r_+}^2}\phi'^2(1) - \frac{\psi^2(1)}{2}e^{-2b\phi'^2(z)/r_+^2}\right]\frac{\phi'(1)}{\left(1 + \frac{4b}{r_+^2}\phi'^2(1)\right)}. \tag{5.22}$$

Now, using the method prescribed by the matching technique[213], we match the solutions Eqs. (5.15a), (5.15b), (5.20) and (5.22) at the intermediate point $z = z_m$. It is very much evident that the matching of the two asymptotic solutions smoothly at $z = z_m$ requires the following four conditions:

$$\mu - \frac{\rho z_m^2}{r_+^2} = \beta(1-z_m) - \frac{\beta}{2}(1-z_m)^2\left[\frac{1-8b\tilde{\beta}^2}{1+4b\tilde{\beta}^2} - \frac{\alpha^2}{2}\frac{e^{-2b\tilde{\beta}^2}}{1+4b\tilde{\beta}^2}\right] \tag{5.23}$$

$$-\frac{2\rho z_m}{r_+^2} = -\beta + \beta(1-z_m)\left[\frac{1-8b\tilde{\beta}^2}{1+4b\tilde{\beta}^2} - \frac{\alpha^2}{2}\frac{e^{-2b\tilde{\beta}^2}}{1+4b\tilde{\beta}^2}\right] \tag{5.24}$$

$$D_+ z_m^{\lambda_+} = \frac{\alpha}{4} + \frac{3\alpha z_m}{4} + \alpha(1-z_m)^2\left[\frac{-15}{64} + \frac{3a}{2} - \frac{\tilde{\beta}^2}{64}\right] \tag{5.25}$$

$$\lambda_+ D_+ z_m^{\lambda_+} = \frac{3\alpha z_m}{4} - 2\alpha z_m(1-z_m)\left[\frac{-15}{64} + \frac{3a}{2} - \frac{\tilde{\beta}^2}{64}\right] \tag{5.26}$$

where we have set $\psi(1) = \alpha$, $-\phi'(1) = \beta$ $(\alpha, \beta > 0)$, $\tilde{\beta} = \frac{\beta}{r_+}$ and $D_- = 0$ [cf.Eq. (5.15b)].

From Eq. (5.24), using Eq. (5.7), we obtain,

$$\alpha^2 = \frac{2z_m}{(1-z_m)}e^{2b\tilde{\beta}^2}\left(1 + \frac{4b\tilde{\beta}^2(3-2z_m)}{z_m}\right)\left(\frac{T_c}{T}\right)^3\left(1 - \frac{T^3}{T_c^3}\right). \tag{5.27}$$

Since, in our entire analysis we intend to keep terms which are only linear in the non-linear parameter, b, Eq. (5.27) can be approximated as,

$$\alpha^2 \approx \frac{2z_m}{(1-z_m)}\left(1+\frac{6b\tilde{\beta}^2(2-z_m)}{z_m}\right)\left(\frac{T_c}{T}\right)^3\left(1-\frac{T^3}{T_c^3}\right). \quad (5.28)$$

Here the quantity T_c may be identified as the critical temperature for condensation and is given by,

$$T_c = \left[\frac{2\rho}{\tilde{\beta}\pi^3}\left(1-\frac{12b\tilde{\beta}^2(1-z_m)}{z_m}\right)\right]^{\frac{1}{3}}. \quad (5.29)$$

Now from Eq. (5.25) and Eq. (5.26) we obtain,

$$D_+ = \frac{(3z_m^2+5z_m)}{4\left(\lambda_+ + (2-\lambda_+)z_m\right)}\left(\frac{1}{z_m}\right)^{\lambda_+}\alpha, \quad (5.30)$$

$$\tilde{\beta} = 8\left[\frac{-15}{64}+\frac{3a}{2}-\frac{(3z_m^2+5z_m)(1+\lambda_+)}{4\left(\lambda_+ + (2-\lambda_+)z_m\right)\left(1-4z_m+3z_m^2\right)}+\frac{(1+6z_m)}{4\left(1-4z_m+3z_m^2\right)}\right]^{\frac{1}{2}}. \quad (5.31)$$

Finally, using Eqs. (5.7), (5.28) and (5.30), near the critical temperature, $T \sim T_c$, we may write the expectation value, $\langle \mathcal{O}_2 \rangle$, of the condensation operator in the following form[4]:

$$\frac{\langle \mathcal{O}_2 \rangle^{\frac{1}{\lambda_+}}}{T_c} = \left(\frac{\pi}{z_m}\right)\left[\frac{(3z_m^2+5z_m)}{4\left(\lambda_+ + (2-\lambda_+)z_m\right)}\right]^{\frac{1}{\lambda_+}}\left[\frac{6z_m}{(1-z_m)}\left(1+\frac{6b\tilde{\beta}^2(2-z_m)}{z_m}\right)\left(1-\frac{T}{T_c}\right)\right]^{\frac{1}{2\lambda_+}}. \quad (5.32)$$

In Eq. (5.32) we have normalized $\langle \mathcal{O}_2 \rangle$ by the critical temperature, T_c, to obtain a dimensionless quantity[213]. Proceeding along the same line of analysis we can also calculate the critical temperature, T_c, and condensation operator, $\langle \mathcal{O}_2 \rangle$, for the holographic superconductors with LNE. The corresponding expressions for the above mentioned quantities are given below:

The critical temperature for condensation:

$$T_c = \left[\frac{2\rho}{\tilde{\beta}\pi^3}\left(1-\frac{3b\tilde{\beta}^2(1-z_m)}{2z_m}\right)\right]^{\frac{1}{3}}. \quad (5.33)$$

Order parameter for condensation:

$$\frac{\langle \mathcal{O}_2 \rangle^{\frac{1}{\lambda_+}}}{T_c} = \left(\frac{\pi}{z_m}\right)\left[\frac{(3z_m^2+5z_m)}{4\left(\lambda_+ + (2-\lambda_+)z_m\right)}\right]^{\frac{1}{\lambda_+}}\left[\frac{6z_m}{(1-z_m)}\left(1+\frac{3b\tilde{\beta}^2(2-z_m)}{4z_m}\right)\left(1-\frac{T}{T_c}\right)\right]^{\frac{1}{2\lambda_+}}. \quad (5.34)$$

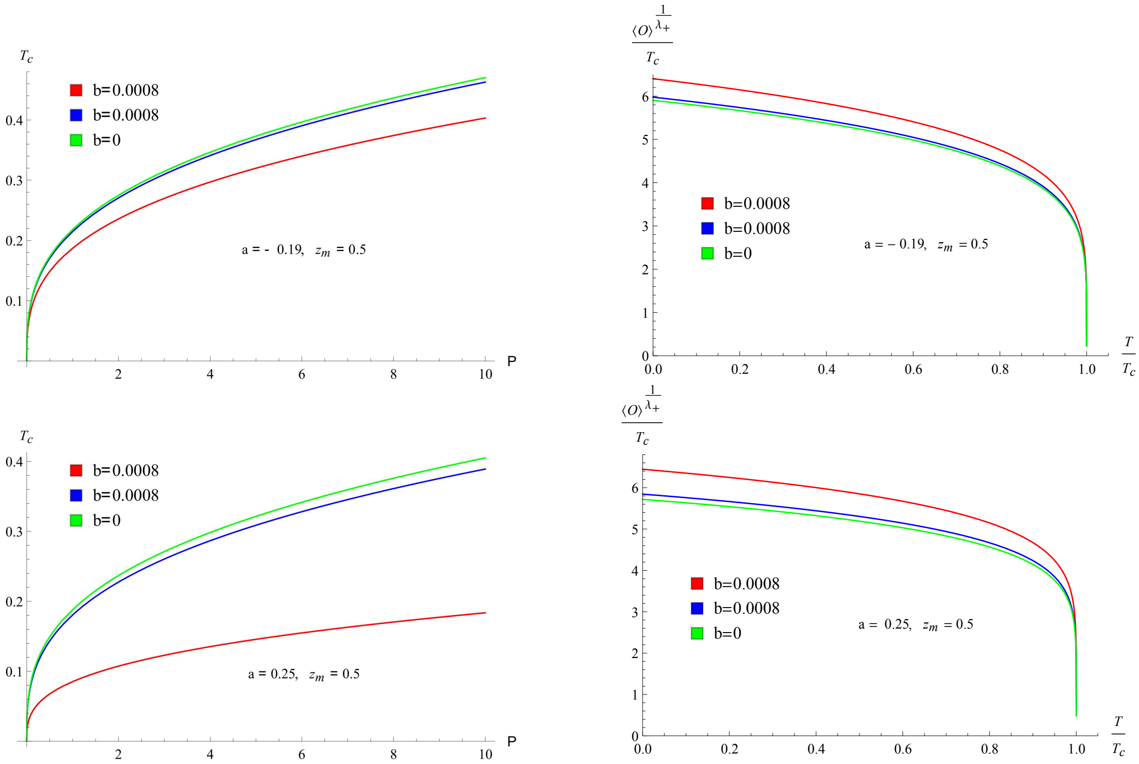

Figure 5.1: Plots of critical temperature ($T_c - \rho$) and normalized condensation operator ($\langle \mathcal{O}_2\rangle^{\frac{1}{\lambda_+}}/T_c - T/T_c$) for different electrodynamic theories. The green, blue and red curves correspond to Maxwell, LNE and ENE, respectively.

In the next section we shall be mainly concerned with the effects of magnetic field on this s-wave holographic superconductor with the two different types of non-linear electrodynamics mentioned earlier. But, before that we would like to make some comments on the results obtained so far. These may be put as follows:

- From Eq. (5.29) (and Eq. (5.33)) it is evident that in order to have a meaningful notion of the critical temperature, T_c, there must have an upper bound to the non-linear coupling parameter, b. The upper bounds corresponding to two non-linear theories are given below:

$$b \leq \begin{cases} \dfrac{z_m(\lambda_+ + 2z_m - \lambda_+ z_m)}{12(1+96a\lambda_+ - 6(32a-13)(\lambda_+ - 1)z_m + 3(32a-5)(\lambda_+ - 2)z_m^2)}, & \text{for ENE} \\[2ex] \dfrac{2z_m(\lambda_+ + 2z_m - \lambda_+ z_m)}{3(1+96a\lambda_+) - 18(32a-13)(\lambda_+ - 1)z_m + 9(32a-5)(\lambda_+ - 2)z_m^2}, & \text{for LNE} \end{cases} \tag{5.35}$$

  Note that, with our choice $z_m = 0.5$ and for fixed values of a, this upper bound is smaller for ENE compared to LNE.

[4]Here we have used the relation $(1-t^3) = (1-t)(1+t+t^2)$ for any arbitrary variable t.

- The critical temperature, T_c, decreases as we increase the values of the non-linear parameter, b (Table 5.1, Table 5.2). This feature is general for the two types of holographic superconductors considered in this paper. It must be remarked that, without any non-linear corrections ($b = 0$) the critical temperature is larger than the above two cases. For example, $T_c = 0.1907\rho^{1/3}$ for $a = 0.2$, $z_m = 0.5$. This suggests the onset of a harder condensation. Another nontrivial and perhaps the most interesting feature of our present analysis is that, for a particular value of the non-linear parameter, b, the value of the critical temperature, T_c, for the holographic condensate with ENE is smaller than that with LNE (Table 5.1, Table 5.2) showing stronger effects of the former on the condensation. It is also noteworthy that similar feature was obtained numerically by the authors of Ref.[236] in the planar Schwarzschild-AdS black hole background.

- The condensation gap for the holographic condensate with non-linear electrodynamics is more than that with Maxwell electrodynamics (Fig. 5.1). On top of that, holographic superconductors with ENE exhibit larger gap compared with that with LNE. This suggests that the formation of the scalar hair is more difficult for the holographic condensate with ENE[236].

- The Gauss-Bonnet parameter (a) also has important consequences in the formation of the holographic condensate. From Table 5.1 and Table 5.2 it is clear that as we increase the value of a the critical temperature for condensation decreases. This means that the increase of a makes the formation of scalar hair difficult. This indeed shows that both a and b has the same kind of influences on the formation of the hair. However, from Fig. 5.2 we observe that T_c decreases more rapidly with b than with a. This clearly suggests that the Born-Infeld parameter (b) modifies the critical temperature more significantly than the Gauss-Bonnet parameter (a).

- The expectation value of the condensation operator, $\langle \mathcal{O}_2 \rangle$, vanishes at the critical point $T = T_c$ and the condensation occurs below the critical temperature, T_c (see the right panel of Fig. 5.1). Moreover, form Eq. (5.32) (and Eq. (5.34)) we observe that $\langle \mathcal{O}_2 \rangle \propto (1 - T/T_c)^{1/2}$ which shows the mean field behaviour of the holographic condensates and signifies that there is indeed a second order phase transition (critical exponent 1/2). This also admires the consistency of our analysis.

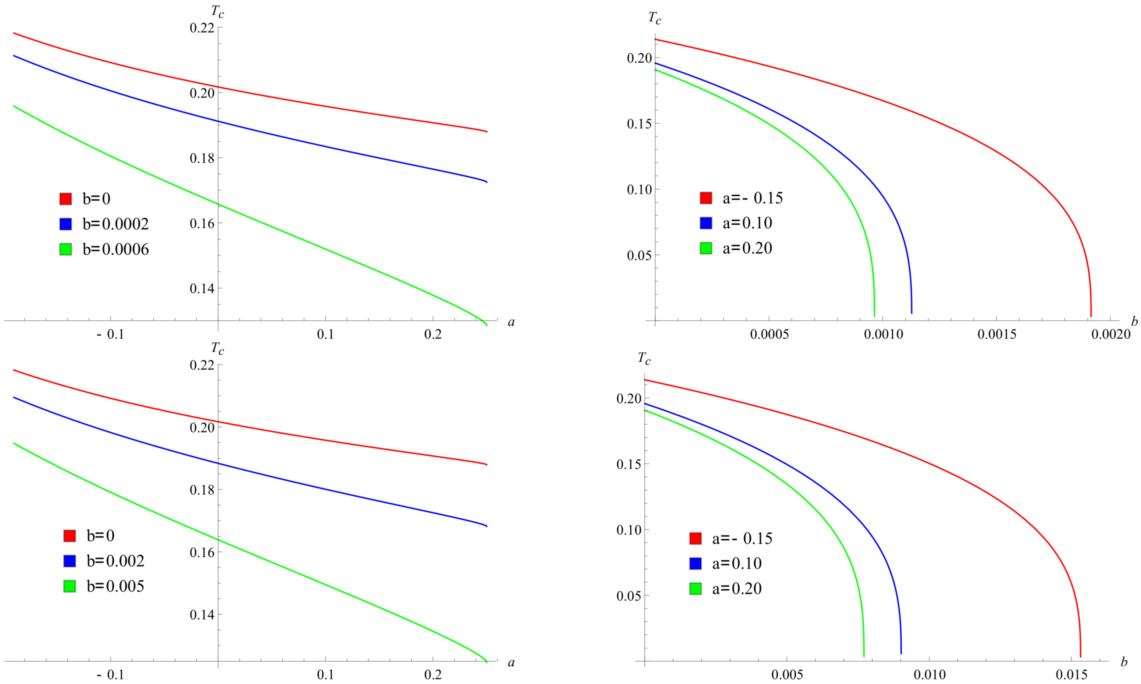

Figure 5.2: Left panel: Variation of T_c with a for different values of b (top for ENE, bottom for LNE); Right panel: Variation of T_c with b for different values of a (top for ENE, bottom for LNE). We have chosen $z_m = 0.5$ and $\rho = 1$.

b	$a = -0.19$		$a = -0.10$	
	ENE	LNE	ENE	LNE
0.0002	0.2113	0.2174	0.2005	0.2081
0.0004	0.2039	0.2165	0.1910	0.2070
0.0006	0.1958	0.2157	0.1805	0.2060
0.0008	0.1871	0.2148	0.1686	0.2049
0.0010	0.1775	0.2139	0.1547	0.2039
0.0012	0.1667	0.2130	0.1377	0.2028
0.0014	0.1542	0.2122	0.1149	0.2016
0.0016	0.1394	0.2113	0.0753	0.2005
0.0018	0.1204	0.2104	—	—
0.0020	0.0923	0.2094	—	—

Table 5.1: Numerical values of coefficients of T_c for different values of the parameters b and $a < 0$.

b	$a = 0.10$		$a = 0.25$	
	ENE	LNE	ENE	LNE
0.0002	0.1834	0.1943	0.1725	0.1861
0.0003	0.1766	0.1936	0.1636	0.1852
0.0004	0.1692	0.1928	0.1537	0.1843
0.0005	0.1610	0.1920	0.1421	0.1834
0.0006	0.1520	0.1913	0.1285	0.1824
0.0007	0.1417	0.1906	0.1110	0.1815
0.0008	0.1296	0.1898	0.0852	0.1805
0.0009	0.1148	0.1890	—	—
0.0010	0.0946	0.1882	—	—

Table 5.2: Numerical values of coefficients of T_c for different values of the parameters b and $a > 0$.

5.4 Magnetic response: Meissner-*like* effect and critical magnetic field

The bizarre properties of superconductors under the influence of external magnetic fields is one of the most significant features these objects possess. In fact, the magnetic response of superconductors helps us to classify them into two distinct classes, namely, type I and type II superconductors. Apart from this, formation of vortex lattice in type II superconductors is also an important aspect that has been taken into consideration over the years. But, the Meissner effect is one of the defining properties of a superconductor that we will describe holographically in this section. It is observed that when immersed in an external magnetic field, ordinary superconductors expel magnetic field lines thereby exhibiting perfect diamagnetism when the temperature is lowered through T_c. This is the Meissner effect[203]. But, for the holographic superconductors in the probe limit we neglect the backreaction of the scalar field on the background geometry. As a result, the superconductors are not able to repel the background magnetic field and the magnetic flux always penetrates the condensates due to the fact that the free energy difference between the normal and the superconducting states are not sufficient to repel these fluxes from such a large volume[194]. Instead, the scalar condensates adjust themselves such that they only fill a finite strip in the plane which reduces the total magnetic field passing through it. In other words, the effect of the external magnetic field is such that it always tries to reduce the condensate away making the condensation difficult to set in. Considering this apparent similarity with the conventional Meissner effect, this holographic phenomena is referred to as *Meissner-like effect*[245].

In order to study the magnetic response of holographic superconductors we add an external static magnetic field in the bulk. According to the gauge/gravity duality, the asymptotic value of the magnetic field in the bulk corresponds to a magnetic field in the boundary field theory, i.e., $B(\mathbf{x}) = F_{xy}(\mathbf{x}, z \to 0)$[245, 246]. Considering the fact that, near the critical magnetic field, B_c, the value of the condensate is small, we may consider the scalar field ψ as a perturbation near B_c. This allows us to adopt the following ansatz for the gauge field and the scalar field[230, 234, 245, 246]:

$$A_\mu = \Big(\phi(z), 0, 0, Bx, 0\Big), \tag{5.36a}$$

$$\psi = \psi(x, z). \tag{5.36b}$$

With the help of Eqs. (5.8), (5.36a) and (5.36b) we may write the equation of motion for the scalar field $\psi(x, z)$ as[230, 234, 239],

$$\psi''(x,z) - \frac{\psi'(x,z)}{z} + \frac{f'(z)}{f(z)}\psi'(x,z) + \frac{r_+^2\phi^2(z)\psi(x,z)}{z^4 f^2(z)} + \frac{1}{z^2 f(z)}\left(\partial_x^2\psi - B^2x^2\psi\right) + \frac{3r_+^2\psi(x,z)}{z^4 f(z)} = 0 \tag{5.37}$$

In order to solve Eq. (5.37) we shall use the method of separation of variables[245, 246]. Let us consider the solution of the following form:

$$\psi(x, z) = X(x)R(z). \tag{5.38}$$

As a next step, we shall substitute Eq. (5.38) into Eq. (5.37). This yields the following equation which is separable in the two variables, x and z.

$$z^2 f(z)\left[\frac{R''(z)}{R(z)} + \frac{R'(z)}{R(z)}\left(\frac{f'(z)}{f(z)} - \frac{1}{z}\right) + \frac{r_+^2\phi^2(z)}{z^4 f^2(z)} + \frac{3r_+^2}{z^4 f(z)}\right] - \left[-\frac{X''(x)}{X(x)} + B^2x^2\right] = 0. \tag{5.39}$$

Note that, the x dependent part of Eq. (5.39) is localized in one dimension. Moreover, this is exactly solvable since it maps the quantum harmonic oscillator. This may be identified as the Schrödinger equation for the corresponding quantum harmonic oscillator with a frequency determined by B[230, 234, 245, 246],

$$-X''(x) + B^2x^2X(x) = C_nBX(x) \tag{5.40}$$

where $C_n = 2n + 1$ ($n = integer$). Since, the most stable solution corresponds to $n = 0$[230, 234, 245], the z dependent part of Eq. (5.39) may be expressed as

$$R''(z) + \left(\frac{f'(z)}{f(z)} - \frac{1}{z}\right)R'(z) + \frac{r_+^2\phi^2(z)R(z)}{z^4 f^2(z)} + \frac{3r_+^2 R(z)}{z^4 f(z)} = \frac{BR(z)}{z^2 f(z)}. \tag{5.41}$$

Now at the horizon, $z = 1$, using Eqs. (5.14) and (5.41), we may write the following equation:

$$R'(1) = \left(\frac{3}{4} - \frac{B}{4r_+^2}\right)R(1). \tag{5.42}$$

On the other hand, at the asymptotic infinity, $z \to 0$, the solution of Eq. (5.41) can be written as

$$R(z) = D_- z^{\lambda_-} + D_+ z^{\lambda_+}. \tag{5.43}$$

Working in the same line as of the previous section, we shall choose $D_- = 0$ here. Near the horizon, $z = 1$, Taylor expansion of $R(z)$ gives

$$R(z) = R(1) - R'(1)(1-z) + \frac{1}{2}R''(1)(1-z)^2 + \cdots \tag{5.44}$$

where we have considered $R'(1) < 0$ without loss of generality.

Now, calculating $R''(1)$ from Eq. (5.41) and using Eq. (5.42) we may write from Eq. (5.44)

$$\begin{aligned} R(z) &= \frac{1}{4}R(1) + \frac{3z}{4}R(1) + (1-z)\frac{B}{4r_+^2}R(1) \\ &+ \frac{1}{2}(1-z)^2\left[3a - \frac{15}{32} + (1-16a)\frac{B}{16r_+^2} + \frac{B^2}{32r_+^4} - \frac{\phi'^2(1)}{32r_+^2}\right]R(1) \end{aligned} \tag{5.45}$$

where in the intermediate step we have used the Leibniz rule [cf. Eq. (5.18)].

Finally, matching the solutions Eqs. (5.43) and (5.45) at the intermediate point $z = z_m$ and performing some simple algebraic steps we arrive at the following equation in B:

$$\begin{aligned} B^2 &+ 2Br_+^2\left[\frac{8\left(\lambda_+ - (\lambda_+ - 1)z_m\right)}{(1-z_m)(\lambda_+ - \lambda_+ z_m + 2z_m)} + (1-16a)\right] \\ &+ \left[\frac{(1+3z_m)\lambda_+ - 3z_m}{2(1-z_m)(\lambda_+ - \lambda_+ z_m + 2z_m)} + \left(3a - \frac{15}{32} - \frac{\phi'^2(1)}{32r_+^2}\right)\right]32r_+^4 = 0 \end{aligned} \tag{5.46}$$

Eq.(5.46) is quadratic in B and its solution is found to be of the following form[239]:

$$\begin{aligned} B = \; & r_+^2\left[\left(\frac{8\left(\lambda_+ - (\lambda_+ - 1)z_m\right)}{(1-z_m)(\lambda_+ - \lambda_+ z_m + 2z_m)} + (1-16a)\right)^2 - \left(\frac{16\left[(1+3z_m)\lambda_+ - 3z_m\right]}{(1-z_m)(\lambda_+ - \lambda_+ z_m + 2z_m)}\right.\right. \\ & \left.\left. + \left(96a - 15 - \frac{\phi'^2(1)}{r_+^2}\right)\right)\right]^{\frac{1}{2}} - r_+^2\left(\frac{8\left(\lambda_+ - (\lambda_+ - 1)z_m\right)}{(1-z_m)(\lambda_+ - \lambda_+ z_m + 2z_m)} + (1-16a)\right). \end{aligned} \tag{5.47}$$

We are interested in determining the critical value of the magnetic field strength, B_c, above which the superconducting phase disappears. In this regard, we would like to consider the case for which $B \sim B_c$. Interestingly, in this case the condensation becomes vanishingly small and we can neglect terms that are quadratic in ψ. Thus, the equation of motion corresponding to the gauge field (Eq. (5.12)), ϕ, may be written as

$$\left(1 + \frac{4bz^4\phi'^2(z)}{r_+^2}\right)\phi''(z) - \frac{1}{z}\phi'(z) + \frac{8bz^3}{r_+^2}\phi'^3(z) = 0. \tag{5.48}$$

In order to solve the above Eq. (5.48) we consider a perturbative solution of the following form:

$$\phi(z) = \phi_0(z) + \frac{b}{r_+^2}\phi_1(z) + \cdots \tag{5.49}$$

where $\phi_0(z)$, $\phi_1(z)$, $\cdots$ are independent solutions and the numbers in the suffices of $\phi(z)$ indicate the corresponding order of the non-linear parameter (b).

Substituting Eq. (5.49) in Eq. (5.48) we obtain

$$\left[\phi_0''(z) - \frac{\phi_0'(z)}{z}\right] + \frac{b}{r_+^2}\left[\phi_1''(z) - \frac{\phi_1'(z)}{z} + 4z^4\phi_0''(z)\phi_0'^2(z) + \frac{8z^3}{r_+^2}\phi_0'^3(z)\right] + \mathscr{O}(b^2) = 0. \tag{5.50}$$

Equating the coefficients of b^0 and b^1 from the l.h.s of Eq. (5.50) to zero we may write

$$b^0: \qquad \phi_0''(z) - \frac{\phi_0'(z)}{z} = 0 \tag{5.51a}$$

$$b^1: \qquad \phi_1''(z) - \frac{\phi_1'(z)}{z} + 4z^4\phi_0''(z)\phi_0'^2(z) + \frac{8z^3}{r_+^2}\phi_0'^3(z) = 0 \tag{5.51b}$$

Now, using the boundary condition Eq. (5.15a) we may write the solution of Eq. (5.51a) as

$$\phi_0(z) = \frac{\rho}{r_+^2}(1 - z^2). \tag{5.52}$$

Using Eq. (5.52) we may simplify Eq. (5.51b) as

$$\phi_1''(z) - \frac{\phi_1'(z)}{z} - 96z^6\left(\frac{\rho}{r_+^2}\right)^3 = 0. \tag{5.53}$$

As a next step, using the asymptotic boundary condition Eq. (5.15a), from Eq. (5.53) we obtain the solution of $\phi_1(z)$ as

$$\phi_1(z) = \frac{2\rho^3}{r_+^6}(z^8 - 1) - \frac{\rho}{r_+^2}(z^2 - 1). \tag{5.54}$$

Substituting Eqs. (5.52) and (5.54) in Eq. (5.49) we finally obtain the solution of the gauge field as

$$\phi(z) = \frac{\rho}{r_+^2}(1 - z^2)\left[1 + \frac{b}{r_+^2} - \frac{2b\rho^2}{r_+^6}(1 + z^4)(1 + z^2)\right]. \tag{5.55}$$

The solution of the gauge field for the holographic superconductor with LNE may be obtained by similar procedure and is given below:

$$\phi(z) = \frac{\rho}{r_+^2}(1 - z^2)\left[1 + \frac{b}{r_+^2} - \frac{b\rho^2}{4r_+^6}(1 + z^4)(1 + z^2)\right]. \tag{5.56}$$

Note that, in the above calculations we have only retained terms which are linear in the non-linear parameter b.

At the asymptotic boundary of the AdS space, $z = 0$, the solution Eq. (5.55) can be approximated as

$$\phi(z) \approx \frac{\rho}{r_+^2}\left[1 + \frac{b}{r_+^2} - \frac{2b\rho^2}{r_+^6}\right] - \frac{\rho}{r_+^2}\left(1 + \frac{b}{r_+^2}\right) z^2. \tag{5.57}$$

Now, comparing Eq. (5.57) with Eq. (5.15a) we may identify the chemical potential, μ, as

$$\mu = \frac{\rho}{r_+^2}\left[1 + \frac{b}{r_+^2} - \frac{2b\rho^2}{r_+^6}\right] \tag{5.58}$$

Near the horizon, $z = 1$, we may write from Eq. (5.48)

$$\phi''(1) = \phi'(1) - \frac{12b}{r_+^2}\phi'^3(1) + \mathscr{O}(b^2) \tag{5.59}$$

Substituting Eq. (5.59) into Eq. (5.16) and using the boundary condition Eq. (5.14) we may write

$$\phi(z) = -\phi'(1)(1-z) + \frac{1}{2}(1-z)^2\left(\phi'(1) - \frac{12b}{r_+^2}\phi'^3(1)\right). \tag{5.60}$$

Matching the solutions Eq. (5.60) and Eq. (5.15a) at the intermediate point z_m and using Eq. (5.58) we can find the following relation:

$$(\beta - 2\eta)(2\eta^3 - 6\beta^3 - \eta) = 0 \tag{5.61}$$

where we have set $-\phi'(1) = \beta$ and $\frac{\rho}{r_+^2} = \eta$. One of the solutions of this quartic equation can be written as

$$\beta = 2\eta$$

which implies

$$\phi'(1) = -\frac{2\rho}{r_+^2}. \tag{5.62}$$

As a final step, substituting Eq. (5.62) into Eq. (5.47) and using Eqs. (5.7) and (5.29) we obtain the critical value of the magnetic field strength as

$$\frac{B_c}{T_c^2} = \pi^2\left(1 + \frac{12b\tilde{\beta}^2(1-z_m)}{\mathscr{C}^2 z_m}\right)\left[\tilde{\beta}\mathscr{C} - \mathscr{M}\left(\frac{T}{T_c}\right)^3\right] \tag{5.63}$$

In a similar manner we can determine the critical value of magnetic strength, B_c, for the holographic superconductor with logarithmic electrodynamics. The expression for B_c for this model is given below:

$$\frac{B_c}{T_c^2} = \pi^2\left(1 + \frac{3b\tilde{\beta}^2(1-z_m)}{2\mathscr{C}^2 z_m}\right)\left[\tilde{\beta}\mathscr{C} - \mathscr{M}\left(\frac{T}{T_c}\right)^3\right]. \tag{5.64}$$

In Eqs. (5.63) and (5.64) the terms $\mathscr{C}$ and $\mathscr{M}$ are given by

$$\mathscr{C} = \left(1 - \frac{\mathscr{A} - \mathscr{M}^2}{\tilde{\beta}^2} x^6\right)^{\frac{1}{2}},$$

$$\mathscr{M} = 1 - 16a + \frac{8(\lambda_+ - (\lambda_+ - 1)z_m)}{(1 - z_m)(\lambda_+ - \lambda_+ z_m + 2z_m)}$$

where

$$\mathscr{A} = \left[96a - 15 + 16\left(\frac{(1 + 3z_m)\lambda_+ - 3z_m}{(1 - z_m)(\lambda_+ - \lambda_+ z_m + 2z_m)}\right)\right].$$

Here, we have normalized B_c by the square of the critical temperature, T_c, such that the critical magnetic field strength, B_c, becomes dimensionless.

In Fig. 5.3 we have plotted Eq. (5.63) (and Eq. (5.64)) as a function of T/T_c. From these plots it is evident that above the critical magnetic field (B_c) the superconductivity is completely destroyed which is also the case for ordinary type II superconductors[203]. From the above analysis we can explain the effects of the Gauss-Bonnet coupling parameter (a) and the non-linear parameter (b) on the holographic condensates. First of all we note that, for fixed values of a the critical magnetic field (B_c) increases with b. Secondly, the critical magnetic field corresponding to the Maxwell case ($b = 0$) is lower than those for ENE and LNE. This indicates that the critical magnetic field strength is higher in presence of the non-linear corrections than the usual Maxwell case. Moreover, this increment is larger for the holographic condensate with ENE than that with LNE. Finally, if we vary a while keeping b constant similar effects are observed, i.e., the critical field strength increases with the Gauss-Bonnet parameter (a). From the preceding discussion we may infer that both the higher order corrections indeed make the condensation harder to form. Moreover, between the two non-linear electrodynamics, the exponential electrodynamics has stronger effects on the formation of the holographic s-wave condensate namely, the formation of the scalar hair is more difficult for holographic superconductor with ENE. This can be explained by noting that the increase in the critical field strength (B_c) tries to reduce the condensate away making the condensation difficult to set in[230, 234, 245].

5.5 Conclusive remarks

In this chapter, considering the probe limit, we have studied a holographic model of superconductor in the higher curvature planar Gauss-Bonnet-AdS black hole background. We have also taken into account two different types of non-linear electrodynamics (exponential and logarithmic non-linear electrodynamics) in the matter Lagrangian which may be considered as higher derivative corrections to the gauge fields in the usual Abelian gauge theory. In addition to that, we have made an analytic investigation on the effects of an external magnetic field on these superconductors.

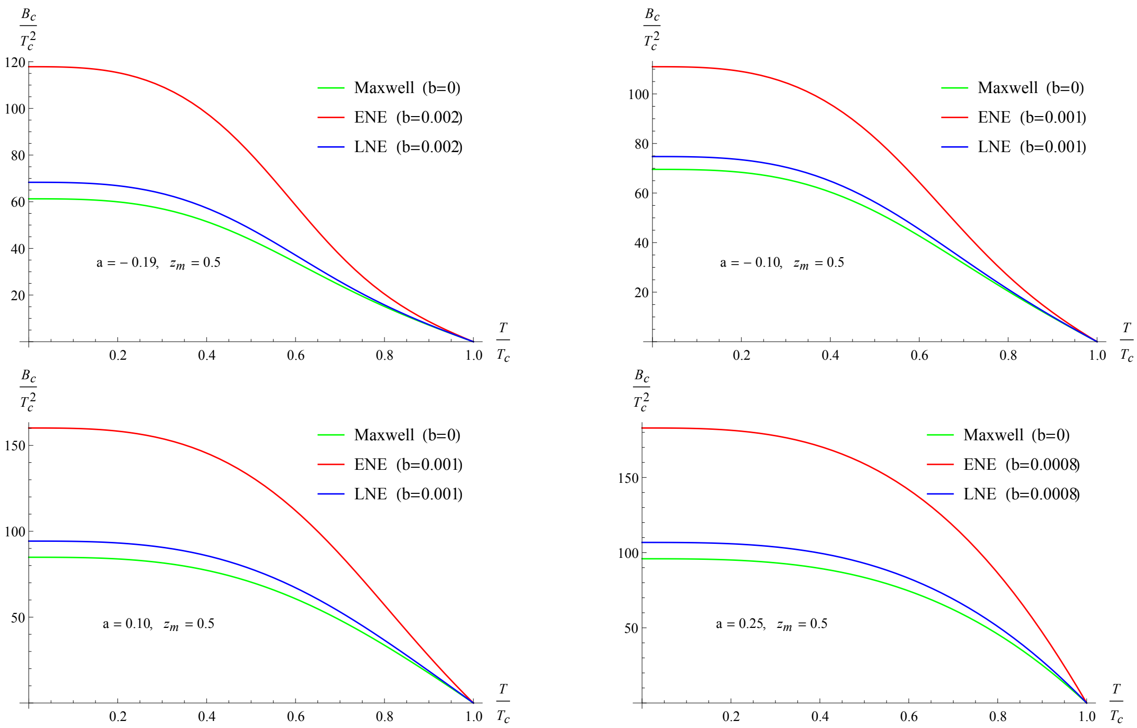

Figure 5.3: Plots of $B_c/T_c^2 - T/T_c$ for different values of a and b.

The primary motivations of the present chapter are to study the effects of several non-linear corrections to the gravity and matter sectors of the action (that describes the holographic superconductivity) on the holographic s-wave condensates both in presence as well as absence of an external magnetic field. Along with these, we aim to make a comparative study among the usual Maxwell electrodynamics and the two NEDs considered in the paper (ENE, LNE) regarding their effects on the formation of holographic condensates. Based on purely analytic methods we have successfully addressed these issues. The main results of our analysis can be put as follows:

- Non-linear electrodynamics has stronger effects on the condensates than the usual Maxwell case. The critical temperature for condensation (T_c) decreases as we increase the values of the non-linear parameter (b) as well as the Gauss-Bonnet coupling parameter (a) (Fig. 5.1, Tables4.1,4.2). Moreover, b modifies the critical temperature more significantly than a (Fig. 5.2). On the other hand, the normalized order parameter ($\langle \mathcal{O}_2 \rangle^{1/\lambda_+}/T_c$) increases with the increase of b and a (Fig. 5.1). This implies that, in the presence of the higher order corrections the formation of the scalar hairs become difficult.

- The variation of the order parameter with temperature, $\langle \mathcal{O}_2 \rangle \propto (1 - T/T_c)^{1/2}$, exhibits a mean-field behavior. Also, the value of the associated critical exponent is 1/2, which further ensures that the holographic condensates indeed undergo a second order phase transition in going from normal to superconducting phase.

- There exists a critical magnetic field B_c above which the superconductivity ceases to exist (Fig. 5.3). This property is similar to that of ordinary type II superconductors [203]. Also, the critical magnetic field strength (B_c) increases as we increase both b and a. The increasing magnetic field strength tries to reduce the condensate away completely making the condensation difficult to form.

- From Figs:5.1,5.3 and Tables4.1 and 4.2 we further observe that for particular parameter values T_c is less in holographic superconductor with ENE than that with LNE whereas, $\langle \mathcal{O}_2 \rangle^{1/\lambda_+}/T_c$ and B_c is more in the previous one. These results suggest that the exponential electrodynamics exhibit stronger effects than the logarithmic electrodynamics.

It is reassuring to note that, similar conclusions were drawn in Ref.[236] where numerical computations were performed in this direction. Our analytic calculations provide further confirmations regarding this issue. However, the novel feature of our present analysis is that we have been able to study the effect of the higher curvature corrections which was not performed explicitly in Ref.[236].

So far we are dealing with holographic superconductors within the framework of the AdS/CFT correspondence. In doing so we are assuming that the scale as well as the Lorentz symmetry of the boundary field theory is well preserved. In other words, we are in the relativistic regime. But, the whole analysis can also be performed in the non-relativistic regime where, although, the scaling symmetry is well preserved the Lorentz symmetry is broken explicitly. However, the scaling symmetry is *anisotropic* in the sense that there exists a certain degree of anisotropy between time and space. As a result of this non-linearity, the boundary field theory is non-relativistic and a suitable modification of the original AdS/CFT duality must be taken into consideration to describe dual holographic models, such as the holographic superconductors, which we are going to explore in the next chapter. From practical points of view, this may be considered as a more *realistic* holographic description of condensed matter systems owing to the inherent non-relativistic nature of ordinary condensed matter systems including superconductors.

Chapter 6

Holographic Lifshitz Superconductors and Their Magnetic Response

6.1 Overview

The emergence of the AdS/CFT correspondence[170]-[175] has opened up new directions in dealing with the strongly correlated systems. Since its discovery, this duality has been extensively used in several areas in physics such as, fluid/gravity correspondence, QCD, and many others[183]-[190]. In addition, its lucidity and wide range of applicability have led physicists to apply this correspondence in order to understand several strongly coupled phenomena of condensed matter physics[189],[193]-[197]. But in many examples of condensed matter physics it is often observed that the behaviors of the systems are governed by *Lifshitz-like fixed points.* These fixed points are characterized by the anisotropic scaling symmetry

$$t \to \lambda^z t, \qquad x^i \to \lambda x^i \qquad (i = 1, 2, \cdots, d). \tag{6.1}$$

The exponent z is called the "dynamical critical exponent", and it describes the degree of anisotropy between space and time[259, 260]. These are non-Lorentz invariant points and hence the systems are non-relativistic in nature[259]-[269].

There have been several attempts to describe these systems holographically using the standard prescriptions of gauge/gravity duality. But due to the nonrelativistic nature of these systems the dual description has been modified and it provides a gravity dual for systems which are realized by nonrelativistic CFTs[261]-[269]. The gravity dual to Lifshitz fixed points is described by the Lifshitz metric[262]:

$$ds^2 = -r^{2z}dt^2 + \frac{dr^2}{r^2} + r^2 dx^i dx_i \tag{6.2}$$

which respects the scale transformation Eq. (6.1) along with an additional scaling $r \to \lambda^{-1}r$. In the limit $z = 1$ it gives the AdS_{d+2} metric. In Eq. (6.2) $dx_i^2 =$

$dx_1^2 + dx_2^2 + \cdots + dx_d^2$ and $r \in (0, \infty)$. Note that, in writing the above metric we set the radius of the AdS space to unity ($L = 1$) in our analysis. On the other hand, in order to describe finite temperature systems, exact black hole solutions in the asymptotically Lifshitz space-time have been found[267]-[269].

In the previous Chapter 4 and Chapter 5 we became familiar with the role of AdS/CFT duality in describing diverse properties of high T_c superconductors. There we studied these holographic models of superconductor by including several higher derivative corrections to the usual Einstein gravity (Gauss-Bonnet correction) as well as in the Maxwell gauge sector (Born-Infeld, exponential and logarithmic corrections). In addition, the response of the holographic superconductors in external magnetic fields was a crucial issue that we exploited in Section 5.4 in Chapter 5. Further extension of these studies show interesting vortex and droplet solutions for these models[245]-[258]. Very recently promising conclusions have been drawn regarding the effects of various corrections to the Einstein-Maxwell sector on the aforementioned solutions[258].

Over the past few years a series of works have been attempted to understand various properties of HS with Lifshitz scaling[276]-[290]. These works demonstrate interesting effects of anisotropy on the characterizing properties of HS with Lifshitz scaling, and also the effects of external magnetic fields on them. Despite these attempts several other important issues have been overlooked which we address in Ref.[287] and based on that paper we intend to discuss these issues in the present chapter. Here we elaborate the following points: *(i)* We compute the vortex and droplet solutions for a Lifshitz HS. This study is motivated by the observation that the anisotropic scaling plays an important role in affecting the behavior of the holographic condensates[276]-[286], and *(ii)* We study the effects of anisotropy on the holographic condensates. We find that the critical parameters of phase transition are affected by the anisotropy in the system.

The present chapter is organized as follows: In Section 6.2 we very briefly review Lifshitz holographic superconductors. We develop the vortex lattice solution for the s-wave superconductors in a Lifshitz black hole background in Section 6.3.1. In Section 6.3.2 we compute a holographic droplet solution for this superconductor in the Lifshitz soliton background. Finally, in Section 6.4 we draw our conclusions.

6.2 Lifshitz holographic superconductors: a brief review

In this chapter we mainly discuss the magnetic response of holographic s-wave Lifshitz superconductors (henceforth HLS). In doing so, we shall not provide the constructional details of these models, rather, we shall build upon important solutions (vortex and droplet solutions) that are most relevant to our study of magnetic responses. Nevertheless, in this section we review very briefly the properties of HLSs that have already been explored in the literature which will be helpful to find motivations for our study of the mentioned solutions in rest of this chapter. The discussion of this section is mainly borrowed from Ref.[283] and consists of three parts as enumerated below.

(1) ***Basic ingredients:*** In order to construct a $(d+1)$-dimensional superconductor which respects the anisotropic scaling symmetry (Eq. (6.1)) a $(d+2)$-dimensional Lifshitz space-time which contains a black hole is considered. This $(d+2)$-dimensional space-time is in fact the gravity dual to this holographic model having the metric Eq. (6.2)[265, 266, 267, 276]. The action that admits this kind of geometry is found to have the following form[266, 267]:

$$\mathscr{S} = \frac{1}{16\pi G}\int d^{d+2}x\sqrt{-g}\left(R - 2\Lambda - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}e^{b\phi}\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu}\right). \tag{6.3}$$

In Eq. (6.3) R is the Ricci scalar, Λ is the cosmological constant, ϕ is a massless scalar field which is coupled to the Abelian gauge field $\mathscr{A}_\mu$ and $\mathscr{F}_{\mu\nu}$ is the Abalian gauge field strength. The finite temperature generalization of Eq. (6.2) which essentially gives a black hole metric with finite temperature (needed to realize the superconducting phase) can be written as[267],

$$ds^2 = -r^{2z}f(r)dt^2 + \frac{dr^2}{r^2 f(r)} + r^2\sum_{i=1}^{d} dx_i^2 \tag{6.4}$$

where

$$f(r) = 1 - \frac{r_+^{z+d}}{r^{z+d}}, \tag{6.5}$$

$$\Lambda = -\frac{(z+d)(z+d-1)}{2}. \tag{6.6}$$

In Eq. (6.3) the auxiliary gauge field $\mathscr{F}_{\mu\nu}$ modifies the asymptotic symmetry of the geometry from AdS to Lifshitz[265, 266, 267, 276]. In addition, to support the black hole geometry (Eq. (6.4)) the backgrounds for ϕ and $\mathscr{F}_{rt}$ are given by[266, 267],

$$e^{b\phi} = r^{-2d} \qquad b^2 = \frac{2d}{z-1} \tag{6.7}$$

$$q^2 = 2(z-1)(z+d) \qquad \mathscr{F}_{rt} = \sqrt{2(z-1)(z+d)}r^{z+d-1}. \tag{6.8}$$

By analytically continuing the black hole metric Eq. (6.4) to the Euclidean sector the Hawking temperature of the black hole is obtained as

$$T = \frac{(z+d)r_+^z}{4\pi} \tag{6.9}$$

which is also the temperature of the dual field theory as prescribed by the holographic dictionary.

In order to obtain a non-zero holographic superconducting condensate the matter action is chosen as

$$\mathscr{S}_M = \int d^{d+2}x\sqrt{-g}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - |(\partial_\mu - iqA_\mu)\psi|^2 - m^2|\psi|^2\right) \tag{6.10}$$

where, $F_{\mu\nu}$ is the $U(1)$ gauge field strength which is different from the auxiliary field $\mathscr{F}_{\mu\nu}$ introduced in Eq. (6.3) and ψ is the complex scalar field with mass m and charge q which eventually condensates below a certain critical temperature, T_c. Note that, this matter action is none other than the Abelian-Higgs action introduced in Refs.[191]-[197].

(2) ***Boundary conditions and methodology:*** The method of obtaining properties of holographic superconductors in the framework of AdS/CFT correspondence was presented in Chapter 4 and Chapter 5. In the non-relativistic framework the same line of analysis is followed for analyzing Lifshitz superconductors. In other words, the equations of motion for the scalar and gauge fields are obtained from the variation of the action Eq. (6.10) under suitable ansatz, namely, $\psi = \psi(r)$ and $A_\mu = (\phi(r), 0, 0, \cdots)$. As a next step, these equations of motion are solved by imposing boundary conditions on the fields in accordance with the holographic dictionary. These are expressed as,

(i) At the asymptotic boundary $r \to \infty$,

$$\psi(r) \approx \frac{C_1}{r^{\triangle_-}} + \frac{C_2}{r^{\triangle_+}}, \tag{6.11}$$

$$\phi(r) \approx \mu - \frac{\rho}{r^{d-z}} \quad (z < d) \qquad \text{and} \qquad \phi(r) \approx \mu - \rho \ln(\xi r) \quad (z = d), \tag{6.12}$$

where μ, ρ, ξ are constants, $\triangle_\pm = \dfrac{(z+d) \pm \sqrt{(z+d)^2 + 4m^2}}{2}$, and the coefficients C_1, C_2 are related to the expectation values of the operators dual to ψ with scaling dimension $\triangle_-$ and $\triangle_+$, respectively.

(ii) At the black hole horizon $r = r_+$, $\phi(r_+) = 0$ and $\psi(r_+)$ is regular.

Now, for spontaneous breaking of $U(1)$ gauge symmetry the coefficient C_1 is set to zero and the mass squared of the scalar field is chosen as $m^2 \geqslant -\dfrac{(z+d)^2}{4}$.

(3) ***Results:*** The results regarding the s-wave HLS may be summarized in the following points.

(i) The critical temperature for condensation (T_c) decreases as the value of the dynamic exponent (z) increases. Also, it decreases with the increase of the conformal dimension ($\triangle_+$) of the dual operator. This indicates the fact that as one increases the anisotropy of space-time the condensate becomes difficult to form.

(ii) The a.c. conductivity of the condensate becomes suppressed as z increases.

(iii) The variation of the order parameter for condensation with temperature is found to be $\langle \mathscr{O} \rangle \sim \left(1 - \dfrac{T}{T_c}\right)^{\frac{1}{2}}$. This is exactly the behavior predicted by the Ginzburg-Landau theory and the critical exponent has the universal value $\dfrac{1}{2}$[203].

To summarize, as the dynamical critical exponent increases, the superconducting phase transition becomes difficult to set in.

6.3 Vortex and droplet solutions in holographic Lifshitz superconductors

Having acquainted with the basic characteristic features of the holographic Lifshitz superconductors, we motivate ourselves to study some other interesting properties of these holographic models in the presence of an external magnetic field in the probe limit. In this chapter we discuss interesting vortex and droplet solutions which are observed in the HLSs.

6.3.1 Holographic vortex solution

In order to construct the holographic vortex lattice solution we consider a HLS in $(2+1)$-dimensions. As a result, we need to take into account a dual gravity theory in $(3+1)$-dimensional Lifshitz space-time. In other words, we are interested in the case with $d = 2$ in all the relevant calculations (see the previous **??**).

The background over which we intend to work is given by the following four dimensional Lifshitz black hole[267, 283]:

$$ds^2 = -\frac{\beta^{2z}}{u^{2z}} f(u)dt^2 + \frac{\beta^2}{u^2}(dx^2 + dy^2) + \frac{du^2}{u^2 f(u)} \tag{6.13}$$

where we have chosen a coordinate $u = \frac{1}{r}$, such that the black hole horizon is at $u = 1$ and the boundary $(r \to \infty)$ is at $u = 0$, for mathematical simplicity. In Eq. (6.13)

$$f(u) = 1 - u^{z+2}, \qquad \beta(T) = \left(\frac{4\pi T}{z+2}\right)^{\frac{1}{z}} \tag{6.14}$$

T being the Hawking temperature of the black hole.

The four dimensional matter action for our model is obtained from Eq. (6.10), after setting $d = 2$, as,

$$\mathscr{S}_M = \int d^4x\sqrt{-g}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - |(\partial_\mu - iqA_\mu)\psi|^2 - m^2|\psi|^2\right). \tag{6.15}$$

The equations of motion for the scalar field (ψ) and the gauge field (A_μ) can be obtained from Eq. (6.15) as

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\partial^\mu\psi\right) - A_\mu A^\mu\psi - m^2\psi - iA^\mu\partial_\mu\psi - \frac{i}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}A^\mu\psi\right) = 0, \tag{6.16}$$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}F^{\mu\nu}\right) = j^\nu \equiv i\left(\psi^*\partial^\nu\psi - \psi(\partial^\nu\psi)^*\right) + 2A^\nu|\psi|^2. \tag{6.17}$$

In order to proceed further, we shall consider the following ansatz for the gauge field[249]:

$$A_\mu = (A_t, A_x, A_y, 0). \tag{6.18}$$

We shall make further assumption that the solutions are stationary, i.e., independent of time t. Using these we may write Eqs. (6.16),(6.17) as a set of coupled differential equations given by

$$\left(u^{3-z}\partial_u \frac{f(u)}{u^{z+1}}\partial_u + \frac{A_t^2}{\beta^{2z} f(u)} - \frac{m^2}{u^{2z}}\right)\psi = \frac{-1}{\beta^2 u^{2z-2}}\left(\delta^{ij}\mathscr{D}_i\mathscr{D}_j\psi\right) \tag{6.19a}$$

$$f(u)\beta^2\partial_u\Big(u^{z-1}(\partial_u A_t)\Big) + u^{z-1} \triangle A_t = \frac{2\beta^2 A_t}{u^{3-z}}\psi^2. \tag{6.19b}$$

where $i, j = x, y$ and $\triangle = \partial_x^2 + \partial_y^2$ is the *Laplacian operator.*

In order to solve the above set of equations we shall invoke the following boundary conditions[249]:

(i) At the asymptotic boundary ($u \to 0$), the scalar field ψ behaves as[283]

$$\psi \sim C_1 u^{\triangle_-} + C_2 u^{\triangle_+} \tag{6.20}$$

where $\triangle_{\pm} = \frac{(z+2)\pm\sqrt{(z+2)^2+4m^2}}{2}$ and the coefficients C_1, C_2 are related to the expectation values of the operators dual to ψ with scaling dimension $\triangle_-$ and $\triangle_+$ respectively. For our analysis we shall always choose the *mass-squared*, m^2, of the scalar field above its lower bound given by $m_{LB}^2 = \frac{-(z+2)^2}{4}$[283]. With this condition both the modes are normalizable and we may choose either one of them as the expectation value of the dual operator while the other behaves as the source. For the rest of our analysis we shall choose $C_1 = 0$. Also, ψ is regular at the horizon, $u = 1$.

(ii) The asymptotic values of the gauge field A_μ give the chemical potential (μ) and the external magnetic field ($\mathscr{B}$) as,

$$\mu = A_t(\vec{x}, u \to 0), \qquad \mathscr{B} = F_{xy}(\vec{x}, u \to 0) \tag{6.21}$$

where $\vec{x} = x, y$. The regularity of the gauge fields demand that $A_t = 0$ and A_i is regular everywhere on the horizon.

We further regard the external magnetic field as the only tuning parameter of our theory. Following this, we assume μ and T of the boundary theory to be fixed and change only $\mathscr{B}$. Considering our model of holographic superconductor analogous to ordinary type-II superconductor, there exists an upper critical magnetic field, $\mathscr{B}_{c_2}$, below which the condensation occurs while above the $\mathscr{B}_{c_2}$ superconductivity breaks down.

As a next step, we define the deviation parameter ϵ such that[249]

$$\epsilon = \frac{\mathscr{B}_{c_2} - \mathscr{B}}{\mathscr{B}_{c_2}}, \qquad \epsilon \ll 1. \tag{6.22}$$

We expand the scalar field ψ, the gauge field A_μ as the following power series in ϵ:

$$\psi(\vec{x}, u) = \epsilon^{1/2}\psi_1(\vec{x}, u) + \epsilon^{3/2}\psi_2(\vec{x}, u) + \cdots, \tag{6.23a}$$

$$A_\mu(\vec{x}, u) = A_\mu^{(0)} + \epsilon A_\mu^{(1)}(\vec{x}, u) + \cdots. \tag{6.23b}$$

From Eqs. (6.22) and (6.23) we may infer the following interesting points:

(i) Since we have chosen $\epsilon \ll 1$, we are in fact very close to the critical point,

(ii) The positivity of the deviation parameter implies that $\mathscr{B}_{c_2}$ is always greater than the applied magnetic field $\mathscr{B}$. This ensures that there is always a non-trivial scalar condensation in the theory that behaves as the order parameter.

Another important point that must be stressed is that, in Eq. (6.23b) $A_\mu^{(0)}$ is the solution to the Maxwell's equation in the absence of scalar condensate ($\psi = 0$). For the rest of our analysis we shall choose the following ansatz:

$$A_\mu^{(0)} = \left(A_t^0(u), 0, A_y^0(x), 0\right) . \tag{6.24}$$

Now matching the coefficients of ϵ^0 on both sides of Eq. (6.19b) we may obtain,

$$A_t^0 = \mu(1-u^{2-z}), \qquad A_x^0 = 0, \qquad A_y^0 = \mathscr{B}_{c_2} x. \tag{6.25}$$

On the other hand using Eqs. (6.23a),(6.25) and using the following ansatz for $\psi_1(\vec{x}, u)$[249]

$$\psi_1(\vec{x}, u) = e^{ipy}\phi(x, u; p) \tag{6.26}$$

where p is a constant, we can write Eq. (6.19a) as,

$$\left(u^{3-z}\partial_u \frac{f(u)}{u^{z+1}}\partial_u + \frac{(A_t^{(0)}(u))^2}{\beta^{2z} f(u)} - \frac{m^2}{u^{2z}}\right)\phi(x, u; p) = \frac{1}{\beta^2 u^{2z-2}}\left[\partial_x^2 + (p - \mathscr{B}_{c_2} x)^2\right]\phi(x, u; p). \tag{6.27}$$

We may solve Eq. (6.27) by using the method of separation of variables[249]. In order to do so we shall separate the variable $\phi(x, u; p)$ as follow:

$$\phi(x, u; p) = \alpha_n(u)\gamma_n(x; p) \tag{6.28}$$

with the separation constant λ_n ($n = 0, 1, 2, \cdots$).

Substituting Eq. (6.28) into Eq. (6.27) we may write the equations for $\alpha_n(u)$ and $\gamma(x; p)$ as

$$u^{2-2z} f(u)\alpha_n^{''}(u) - \left[\frac{(z+1)f(u)}{u^{2z-1}} + (z+2)u^{3-z}\right]\alpha_n^{'}(u) - \frac{m^2}{u^{2z}}\alpha(u) + \frac{(A_t^{(0)})^2}{\beta^{2z} f(u)}\alpha(u) = \frac{\lambda_n \mathscr{B}_{c_2}}{\beta^2 u^{2z-2}}\alpha_n(u), \tag{6.29a}$$

$$\left(\partial_X^2 - \frac{X^2}{4}\right)\gamma_n(x; p) = \frac{\lambda_n}{2}\gamma_n(x; p). \tag{6.29b}$$

where we have identified $X = \sqrt{2\mathscr{B}_{c_2}}\left(x - \frac{p}{\mathscr{B}_{c_2}}\right)$. Following Ref.[245], we can write the solutions of Eq. (6.29b) in terms of *Hermite functions*, H_n, with eigenvalue $\lambda_n = (2n+1)$ as

$$\gamma_n(x; p) = e^{-X^2/4} H_n(X). \tag{6.30}$$

Note that we have considered λ_n to be an odd integer. Since the Hermite functions decay exponentially with increasing X, which is the natural physical choice, our consideration is well justified[245]. Moreover, $\lambda_n = 1$ corresponds to the only physical solution for our analysis. Thus we shall restrict ourselves to the $n = 0$ case. With this choice Eq. (6.30) can be written as

$$\gamma_0(x;p) = e^{-X^2/4} \equiv \exp\left[-\frac{\mathscr{B}_{c_2}}{2}\left(x - \frac{p}{\mathscr{B}_{c_2}}\right)^2\right]. \tag{6.31}$$

From the above analysis it is clear that λ_n is independent of the constant p. Therefore, a linear combination of the solutions $e^{ipy}\alpha_0(u)\gamma_0(x;p)$ with different values of p is also a solution to the EoM for ψ_1. Thus, following this proposition, we obtain

$$\psi_1(\vec{x}, u) = \alpha_0(u) \sum_{l=-\infty}^{\infty} c_l e^{ip_l y}\gamma_0(x;p_l). \tag{6.32}$$

At this point of discussion it is interesting to note that Eq. (6.32) is very similar to the expression for the order parameter of the Ginzburg-Landau (G-L) theory of type-II superconductors in the presence of a magnetic field[203]

$$\psi_{G-L} = \sum_l c_l e^{ip_l y}\exp\left[-\frac{(x - x_l)^2}{2\xi^2}\right] \tag{6.33}$$

where $x_l = \dfrac{k\Phi_0}{2\pi\mathscr{B}_{c_2}}$, Φ_0 being the *flux quanta* and ξ is the *superconducting coherence length.* Comparing Eq. (6.33) with Eq. (6.31) we may obtain the following relation between the critical magnetic field and the coherence length as

$$\mathscr{B}_{c_2} \propto \frac{1}{\xi^2} \tag{6.34}$$

which is indeed in good agreement with the result of the G-L theory[203].

We obtain the *vortex lattice solution* by appropriately choosing c_l and p_l. In order to do so, we shall assume periodicity both in the x and y directions characterized by two arbitrary parameters a_1 and a_2. The periodicity in the y direction may be expressed as

$$p_l = \frac{2\pi l}{a_1\xi}, \qquad l \in \mathbb{Z}. \tag{6.35}$$

Using Eqs. (6.34),(6.35) we may rewrite Eq. (6.31) for different values of l as

$$\gamma(x,y) = \sum_{l=-\infty}^{\infty} c_l \exp\left(\frac{2\pi i l y}{a_1\xi}\right)\exp\left[-\frac{1}{2\xi^2}\left(x - \frac{2\pi l\xi}{a_1}\right)^2\right] \tag{6.36}$$

where the coefficient c_l may be chosen as

$$c_l = \exp\left(\frac{-i\pi a_2 l^2}{a_1^2}\right). \tag{6.37}$$

As a next step, we rewrite Eq. (6.36) by using the *elliptic theta function*, $\vartheta_3(v,\tau)$,[1] as

$$\psi_1(\vec{x},u) = \alpha_0(u)\mathrm{exp}\left(\frac{-x^2}{2\xi^2}\right)\vartheta_3(v,\tau). \tag{6.38}$$

where v and τ may be identified as

$$v = \frac{y-ix}{a_1\xi}, \qquad \tau = \frac{2\pi i - a_2}{a_1^2}. \tag{6.39}$$

Following Refs.[252],[258] and using the pseudo-periodicity of $\vartheta_3(v,\tau)$ we see that the function $\sigma(\vec{x}) \equiv \left|\mathrm{exp}\left(\frac{-x^2}{2\xi^2}\right)\vartheta_3(v,\tau)\right|^2$ represents a vortex lattice in which the fundamental region is spanned by the following two lattice vectors

$$\vec{v}_1 = a_1\xi\partial_y, \qquad \vec{v}_2 = \frac{2\pi\xi}{a_1}\partial_x + \frac{a_2}{a_1}\partial_y. \tag{6.40}$$

We may put forward the main results of this subsection as follows:

(i) From Eq. (6.38) it is observed that the vortex solution does not depend upon the dynamic exponent z. This suggests that, whether the boundary field theory is relativistic or non-relativistic, the vortex structure remains the same. Although, it is interesting to note that the exponent z may have non-trivial effects on the condensation of the scalar field as is evident from Eq. (6.29a).

(ii) Eq. (6.38) also suggests that the structure of the vortex lattice is indeed controlled by the superconducting coherence length, ξ. Moreover, the solution has a *Gaussian profile* along the x direction. As the coherence length decreases the lattice structure gradually dies out. This behavior is similar to that of ordinary type-II superconductors[203].

6.3.2 Holographic droplet solution

While discussing holographic superconducting phase transitions in Chapter 4, Chapter 5 and Section 6.3.1 we have considered the usual thermal phase transitions. In these cases the superconducting phases appear below certain critical temperatures after local $U(1)$ symmetry breaking in the normal conducting phases. In this regard, the normal conducting phase is mimicked by a charged black hole without scalar hair (in the case of s-wave superconductors in which we are interested in) and the superconducting phase is indeed a hairy black hole.

But, there is another kind of phase transition that leads to holographic superconductivity. In this case the normal state is an insulating phase which is described holographically by solitons (AdS or non-AdS)[80, 253]. This insulator/superconductor phase transition may be viewed as a quantum phase transition as the solitons do

[1] Here we use the following definition of the elliptic theta function: $\vartheta(v,\tau) = \sum_{l=-\infty}^{\infty} \exp\left(2i\pi v l + i\pi\tau l^2\right)$.

not possess temperature and unlike black holes there is no such notion as horizon. Before presenting the geometrical details of Lifshitz soliton, let us make ourselves familiar with the holographic insulating phase and insulator/superconductor phase transition. These are the main themes of this section and are useful for finding the desired droplet solution.

There are evidences that in the phase diagram of high-T_c cuprate superconductors and layered organic conductors an insulating phase with antiferromagnetic order is located near the superconducting phase[253]. Interestingly, the soliton geometry realizes this kind of insulating phase[80, 253]. In fact, the spectrum of fluctuations over the AdS soliton has a temperature independent mass gap which resembles the insulating phase. Mathematically, if one considers a gauge theory on a manifold $\mathscr{Y} \times S^1$, with $\mathscr{Y}$ being a spatial manifold, then a mass gap implies that the correlation function $\langle \mathscr{O}(y,z)\mathscr{O}'(y',z)\rangle$ vanishes exponentially when $|y-y'| \to \infty$. Here, y and z are coordinates in $\mathscr{Y}$ and S^1, respectively[80]. Moreover, this phase can be viewed as a confining vacuum state that decays into the black hole via a (Hawking-Page) phase transition. This is dual to a confinement/deconfinement phase transition[80]. In Ref.[269] it is shown that similar kind of phase transition takes place in asymptotically Lifshitz space-times. This indicates the fact that an insulator/superconductor phase transition is also possible under the framework of non-relativistic holographic duality. As a matter of fact, this phenomena has been studied explicitly in Ref.[283].

The insulator/superconductor phase transition is realized in the CFT language as a phase transition in which a large enough $U(1)$ chemical potential, μ, overcomes the mass gap related to the scalar field (ψ) in the theory. This mechanism allows ψ to condensate above a critical value, μ_c. The mechanism of condensate formation is easier than that in the black hole since there is no horizon in the geometry.

In this section, considering the insulator/superconductor phase transition, we extract the holographic droplet solution in the Lifshitz soliton background. To achieve this we consider a planar Lifshitz soliton in 5-dimensions of the following form[269, 283]:

$$ds^2 = -r^2dt^2 + r^2(dx^2+dy^2) + \frac{dr^2}{r^2 f(r)} + r^{2z} f(r) d\chi^2 \tag{6.41}$$

where

$$f(r) = \left(1 - \frac{1}{r^{z+3}}\right). \tag{6.42}$$

This soliton solution is obtained by performing a double Wick rotation of the 5-dimensional Lifshitz black hole solution[269, 283]: $dt \to id\chi$, $d\chi \to idt$. In doing so, the temporal anisotropy in the boundary space-time (Eq. (6.1)) is removed and the dual boundary space-time possesses only spatial anisotropy: $t \to \lambda t$, $x_i \to \lambda x_i$, $\chi \to \lambda^z \chi$. Note that, in this geometry (Eq. (6.41)) the spatial direction χ is compactified to a circle and has a periodicity $\chi = \chi + \pi$. In fact, as we compactify one of the space directions in this asymptotic Lifshitz spacetime, the Lifshitz soliton is dual to a Scherk-Schwarz compactification of a 4-dimensional conformal gauge theory. This soliton geometry with an extra compactified spatial direction, precisely generates this mass gap resembling an insulating phase. The dual field theory is thus $(2+1)$-dimensional as is the case in the high-T_c superconductors. The Lifshitz soliton geometry looks like a cigar in the (r,χ) directions having a tip at $r=r_0$, say.

In our search for droplet solution it will be more convenient to work in polar coordinates, $x = \rho\ sin\theta,\ y = \rho\ cos\theta$ [254, 257]. With this choice of coordinates Eq. (6.41) becomes

$$ds^2 = -r^2dt^2 + r^2(d\rho^2 + \rho^2 d\theta^2) + \frac{dr^2}{r^2 f(r)} + r^{2z} f(r) d\chi^2. \tag{6.43}$$

To obtain non-zero condensate we consider Maxwell-scalar action in 5-dimensions as the matter action of our theory[191]-[197]:

$$\mathscr{S}_M = \int d^5x\sqrt{-g}\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - |(\partial_\mu - iqA_\mu)\psi|^2 - m^2|\psi|^2\Big). \tag{6.44}$$

In the probe limit we shall choose the following ansatz for the gauge field close to the critical point of phase transition ($\mu \sim \mu_c,\ \psi \sim 0$)

$$A = \mu_c dt + \frac{1}{2}\mathscr{B}\rho^2 d\theta \tag{6.45}$$

where μ is the chemical potential and $\mathscr{B}$ is the constant external magnetic field related to the vector potential.

We derive the equation of motion for the scalar field (ψ) by varying the action Eq. (6.44) w.r.t. ψ and the result is

$$\partial_r^2 F(t,r) + \left(\frac{f'(r)}{f(r)} + \frac{(z+4)}{r}\right)\partial_r F(t,r) - \frac{\partial_t^2 F(t,r)}{r^4 f(r)} + \frac{2i\mu_c}{r^4 f(r)}\partial_t F(t,r)$$
$$+\left[\frac{\partial_\chi^2 H(\chi)}{r^{2z+2}f^2(r)H(\chi)} - \frac{m^2}{r^2 f(r)} - \frac{\mathscr{B}^2\rho^2}{4r^4 f(r)} + \frac{\mu_c^2}{r^4 f(r)} + \frac{\partial_\rho\left(\rho\partial_\rho U(\rho)\right)}{r^4 f(r)U(\rho)\rho}\right] F(t,r) = 0. \tag{6.46}$$

In deriving Eq. (6.46) we have used Eq. (6.45) and considered the following ansatz

$$\psi(t,r,\chi,\rho) = F(t,r)H(\chi)U(\rho). \tag{6.47}$$

Now, applying the method of separation of variables we finally obtain the following three equations:

$$\frac{1}{\rho}\partial_\rho\left(\rho\partial_\rho U(\rho)\right) - \frac{1}{4}\mathscr{B}^2\rho^2 U(\rho) = -k^2 U(\rho), \tag{6.48a}$$

$$\partial_\chi^2 H(\chi) = -\lambda^2 H(\chi), \tag{6.48b}$$

$$\partial_r^2 F(t,r) + \left(\frac{f'(r)}{f(r)} + \frac{(z+4)}{r}\right)\partial_r F(t,r) - \frac{\partial_t^2 F(t,r)}{r^4 f(r)}$$
$$+\frac{2i\mu_c}{r^4 f(r)}\partial_t F(t,r) + \frac{1}{r^4 f(r)}\left[\mu_c^2 - m^2r^2 - k^2 - \frac{\lambda^2}{f(r)r^{2z-2}}\right]F(t,r) = 0, \tag{6.48c}$$

where λ and k are some arbitrary constants.

Eq. (6.48b) has the solution of the form

$$H(\chi) = \exp(i\lambda\chi) \tag{6.49}$$

which gives $\lambda = 2n,\ n \in \mathbb{Z}$, owing to the periodicity of $H(\chi)$ mentioned earlier.

Eq. (6.48a) is similar to the equation of a *harmonic oscillator* with $k^2 = l|\mathscr{B}|$, $l \in \mathbb{Z}^+$. We shall expect that the lowest mode of excitation ($n = 0,\ l = 1$) will be the first to condensate and will give the most stable solution after condensation[254, 257].

At this point, let us discuss one of the main results of this section. From Eq. (6.48a) we observe that it has the following solution

$$U(\rho) = \exp\left(\frac{-|\mathscr{B}|\rho^2}{4}\right). \tag{6.50}$$

This suggests that for any finite magnetic field, the holographic condensate will be confined to a finite circular region. Moreover, if we increase the magnetic field this region shrinks to its size and for a large value of the magnetic field this essentially becomes a point at the origin with a nonzero condensate. This is precisely the holographic realization of a superconducting droplet.

As a next step, we shall be interested in solving Eq. (6.48c) in order to determine a relation between the critical parameters (μ_c and $\mathscr{B}$) in this insulator/superconductor phase transition. In order to do so, we shall further define $F(t,r) = e^{-i\omega t}R(r)$. With this definition we may rewrite Eq. (6.48c) as,

$$R''(u) + \left(\frac{f'(u)}{f(u)} - \frac{z+2}{u}\right)R'(u) + \frac{1}{f(u)}\left(\mu_c^2 - \mathscr{B} - \frac{m^2}{u^2}\right)R(u) = 0 \tag{6.51}$$

where $u = \frac{1}{r}$, and we have put $\omega = 0$ since we are interested in perturbations which are marginally stable[254, 257]. Here 'prime' denotes derivative w.r.t. u.

We choose a trial function $\Lambda(u)$ such that

$$R(u \to 0) \sim \langle \mathscr{O}_{\triangle_+} \rangle u^{\triangle_+} \Lambda(u) \tag{6.52}$$

where $\triangle_\pm = \frac{(z+3) \pm \sqrt{(z+3)^2 + 4m^2}}{2}$, $m^2_{LB} = \frac{-(z+3)^2}{4}$ [283], and $\Lambda(0) = 1$, $\Lambda'(0) = 0$. Note that, we have identified C_2 in Eq. (6.20) as the expectation value of the condensation operator, $\langle \mathscr{O}_{\triangle_+} \rangle$.

Substituting Eq. (6.52) into Eq. (6.51) we finally get,

$$\Big[\mathscr{P}(u)\Lambda'(u)\Big]' + \mathscr{Q}(u)\Lambda'(u) + \Gamma\mathscr{R}(u)\Lambda(u) = 0 \tag{6.53}$$

where $\Gamma = \left(\mu_c^2 - \mathscr{B}\right)$, and

$$\mathscr{P} = \left(1 - u^{z+3}\right)u^{2\triangle_+ - z - 2} \tag{6.54a}$$

$$\mathscr{Q} = \Big[\triangle_+\left(\triangle_+ - 1\right)\left(1 - u^{z+3}\right) - m^2 - \triangle_+\left(z + 2 + u^{z+3}\right)\Big]u^{2\triangle_+ - z - 4} \tag{6.54b}$$

$$\mathscr{R} = u^{2\triangle_+ - z - 2}. \tag{6.54c}$$

Interestingly, Eq. (6.53) is a standard *Sturm-Liouville* eigenvalue equation. Thus, we may write the eigenvalue, Γ, by using the following formula[216]

$$\Gamma = \frac{\int_0^1 du \Big(\mathscr{P}(u)(\Lambda'(u))^2 + \mathscr{Q}(u)\Lambda^2(u) \Big)}{\int_0^1 du \mathscr{P}(u)\Lambda^2(u)} = \Gamma(\alpha, z, m^2) \tag{6.55}$$

where we have chosen $\Lambda(u) = 1 - \alpha u^{\triangle_+}$. Thus we may argue that, unlike the case of usual holographic superconductors[257], the quantity $\Gamma = \left(\mu_c^2 - \mathscr{B}\right)$ depends on the dynamic critical exponent (z). Therefore we may conclude that the relation between the parameters of the phase transition depend upon the anisotropic scaling. In the Tables 6.1,6.2,6.3 below we have shown the non-trivial dependence of Γ on z.

m^2	-3.0	-2.0	-1.0	1.0	2.0	3
Γ	2.41947	5.51156	7.59806	11.1513	12.7798	14.3487

Table 6.1: Variation of Γ for $z = \frac{1}{2}$ ($m_{LB}^2 = -3.0625$)

m^2	-4.5	-3.5	-2.5	-1.5	1.5	2.5	3.5	4.5
Γ	4.96028	7.50645	9.59179	11.479	16.5666	18.1496	19.6951	21.2097

Table 6.2: Variation of Γ for $z = \frac{3}{2}$ ($m_{LB}^2 = -5.06$)

m^2	-7.0	-5.0	-3.0	-1.0	1.0	3.0	5.0	7.0
Γ	5.61026	10.5782	14.4483	17.9369	21.2117	24.3448	27.3750	30.3261

Table 6.3: Variation of Γ for $z = \frac{5}{2}$ ($m_{LB}^2 = -7.5625$)

6.4 Conclusive remarks

In this chapter we have focused our attention to the study of a holographic model of s-wave superconductor with Lifshitz scaling in the presence of external magnetic field by using the gauge/gravity duality. Working in the probe limit, we have constructed vortex and droplet solutions for our holographic model by considering a Lifshitz black hole and a Lifshitz soliton background, respectively. Unlike the AdS/CFT holographic superconductors there is a non-trivial dynamic exponent in the theory which is responsible for an anisotropy between the temporal and the spatial dimensions of the space-time resulting certain noticeable changes of the properties of the superconductor[276]-[290]. Also, due to the non-relativistic nature of the filed theory, the model is governed by the AdS/NRCFT correspondence[261]-[269].

The primary motivation of the present study is to verify the possibility of vortex and droplet solutions, which are common to the usual holographic superconductors described by the AdS/CFT correspondence[245]-[249],[252],[257, 258], for this class of holographic superconductors as well as to consider the effects of anisotropy on these solutions. Based on purely analytic methods we have been able to construct these solutions. Our analysis shows that, although, the anisotropy has no effects on the vortex lattice solutions, it may have a non-trivial effect on the formation of holographic condensates. Also, a close comparison between our results and those of the Ginzburg-Landau theory reveals the fact that the upper critical magnetic field ($\mathscr{B}_{c_2}$) is inversely proportional to the square of the superconducting coherence length (ξ). This allows us to speculate the behavior of $\mathscr{B}_{c_2}$ with temperature although this requires further investigations which is expected to be explored in the future. On the other hand, based on the method of separation of variables, we have been able to model a holographic droplet solution by working in a Lifshitz soliton background and considering insulator/superconductor phase transition. Our analysis reveals that a holographic droplet is indeed formed in the $\rho - \theta$ plane with a non-vanishing condensate. Also, this droplet grows in size until it captures the entire plane when the external magnetic field $\mathscr{B} \to 0$. Interestingly, it is observed that the anisotropy does not affect the droplet solution. On top of that, we have determined a relation between the critical parameters of the phase transition by using the Sturm-Liouville method[216]. Interestingly, this relation is solely controlled by the dynamic exponent (z) which in turn exhibits the effects of anisotropy on the condensate (cf. Eq. (6.55)).

Chapter 7

Summary and Outlook

In the previous five chapters, Chapter 2 through Chapter 6, we have presented in details the works on which the present thesis is based on. Now, let us summarize the entire analysis presented so far, and mention some of the future directions in which we may further make some progress. At this stage we must mention that the motivation of the present thesis was to investigate certain aspects of black holes in the presence of non-linearity. In the major part of the thesis (Chapter 2–Chapter 5) we considered non-linearity in the gauge and/or gravity sector of the usual Einstein-Maxwell gravity in anti-de Sitter (AdS) space-time. On the other hand, in Chapter 6, we also considered gravity theories that arise from non-linear (anisotropic) scaling symmetry of space and time, namely, the Lifshitz gravity theories. While Chapter 2 and Chapter 3 contain thermodynamic aspects of several black holes subject to non-linear corrections, the rest of the chapters deal with non-linear aspects of holographic s-wave superconductors in the framework of gauge/gravity duality (relativistic and non-relativistic) in which black holes play a pivotal role.

Let us summarize the results of Chapter 2. In this chapter we have applied a fundamental scheme of ordinary, text-book thermodynamics in order to study phase transition phenomena in charged AdS black holes with non-linear Born-Infeld correction. This is the well known *Ehrenfest's scheme* which plays an important role in determining the nature of the phase transition in ordinary thermodynamics[124]-[128]. In order to maintain compatibility with the laws of black hole mechanics[59] we have suitably modified the thermodynamic variables in our study. Note that, we have applied this scheme after carefully analyzing the temperature-entropy $(T-S)$ plot which is devoid of any discontinuity indicating the absence of any first order transition. This observation encouraged us to check the validity of two Ehrenfest's equations at the critical points as is done in standard thermodynamic systems. Based on our analysis, we have determined the order of the phase transition in the Born-Infeld-AdS (BI-AdS) black hole which is found to be of second order. Here the transition occurs from a lower mass black hole with negative specific heat to a higher mass black hole with positive specific heat thereby attaining stability. In order to check the validity of the *Ehrenfest's scheme* in describing phase transitions in black holes we have made an extensive study of the phase transition in the framework of the *thermodynamic state space geometry.* We have found that our approach is compatible with the latter. This is reassured by the observation that the points of divergence of the heat capacity are identical with those of the *Ruppeiner scalar curvature.*

We have continued our study of black hole thermodynamics in Chapter 3. Here, working in the same line of Chapter 2, we qualitatively describe the thermodynamic phase structure and stability of a third order Lovelock-Born-Infeld-AdS (LBI-AdS) black hole. Notably, this charged black hole solution is derived from the modified Einstein-Maxwell theory in which both gauge and gravity sector have been modified to contain Born-Infeld and third order curvature corrections, respectively. We have further extended our investigation by studying the critical behavior of the mentioned black hole in the vicinity of the critical points. We also have computed the static critical exponents, and checked the validity of the static scaling laws and static scaling hypothesis explicitly. These critical exponents form a unique set of values different from any known thermodynamic system. Comparing with earlier observations[166]-[168] we found that all the charged AdS black holes belong to the same universality class.

In the remaining three chapters of the thesis we have devoted ourselves in the study of the effects of non-linear gauge as well as gravity corrections on some important aspects of holographic s-wave superconductors. These are phenomenological models of high-T_c superconductors and are studied in the framework of gauge/gravity dualities. As a matter of fact, the gauge/gravity duality has appeared to be a valuable tool to describe strongly interacting field theories to which the high-T_c superconductors belong.

In Chapter 4, inspired by the fundamental role of the AdS/CFT duality to describe several properties of holographic superconductors[193], we have analytically studied the effects of Born-Infeld corrections (to the gauge fields in the Abelian-Higgs sector of the s -wave holographic superconductors) on the holographic superconducting phase transition. Due to the presence of non-linearity in the theory, it has been observed that the analysis is extremely non-trivial, and a genuine mathematical approach has been taken into consideration based on which we have been able to determine the critical temperature (T_c) and the order parameter ($\langle\mathcal{O}\rangle$) of phase transition successfully. Our analysis reveals that the higher derivative Born-Infeld correction indeed makes the condensate formation harder as with increasing Born-Infeld parameter (b) T_c and $\langle\mathcal{O}\rangle$ decreases and increases, respectively. Moreover, using the *Sturm-Liouville eigenvalue method*[216], we have been able to compute T_c as a function of charge density (ρ). In addition, the critical exponent associated with $\langle\mathcal{O}\rangle$ has been found to be 1/2 which describes a second order superconducting phase transition, and is the universal feature of a mean field theory.

In Chapter 5 we have explored the properties of s-wave holographic Gauss-Bonnett superconductors in the presence of two Born-Infeld-like higher derivative corrections to the gauge field, namely, the logarithmic and exponential corrections. By considering this particular model we have been able to demonstrate the effects of both gauge and gravity corrections on the superconducting phase transition. We observed that these corrections reduce the critical temperature for condensation and subsequently the condensation formation becomes harder. On top of that, we also study the effect of an external magnetic field on the condensate. Our analysis has revealed that a *Meissner-like* effect indeed takes place which tries to reduce the condensate away which is not favorable for condensate formation. Using the *Matching method*[213], we have also computed analytically the order parameter and the critical magnetic field for the condensate which are found to increase with increasing non-linearity. Thus, the presence of these corrections make it difficult for the superconducting phase transition to take place. Along with these, we have made a comparative study of the effects of two different kinds of non-linear corrections and found that the gauge corrections have more significant effects on the condensate formation than the curvature correction. However, between two gauge corrections the exponential correction exhibits stronger effects than the logarithmic correction.

In the penultimate Chapter 6 we have studied the magnetic response of holographic s-wave Lifshitz superconductors. These are in fact prototype holographic models of strongly interacting non-relativistic field theories where Lorentz symmetry is violated explicitly, although the scaling symmetry remains preserved. Here we have investigated the effects of the dynamic critical exponent (z), a measure of anisotropy between space and time which is inherent to the non-relativistic field theory, on the vortex and droplet solutions that we have obtained analytically. Surprisingly, we have observed that the anisotropy has no effect on these solutions which helps us to conclude that these solutions are independent of the nature of the field theory, i.e., whether the corresponding field theory is relativistic or non-relativistic, the solutions are the same. However, we have found non-trivial dependence of the critical parameters of this holographic phase transition on the dynamic exponent.

Finally, as we proceed towards the end of the thesis, we would like to discuss several implications of the works presented here. We present these in the following points:

(i) From our discussion on the thermodynamic aspects of black holes, the relation between the laws of black hole mechanics and that of ordinary thermodynamics becomes more transparent. It also helps us to reestablish the fact that black holes can be described by mean field approximation. The compatibility of the *Ehrenfest's scheme* with the thermodynamic state space geometry approach and the study of critical phenomena in black holes seems to support this argument. Further analysis in this direction may put this connection in a firm basis. In this regard, the critical exponents associated with the correlation length and correlation function, that we only speculated, may be determined using an alternative scheme based on the Renormalization Group approach in which the thermodynamic state space geometry may play an important role.

(ii) The surprising connection between gravity theories and strongly interacting filed theories has played promising role to understand strongly coupled condensed matter theories in ample details. In fact, based on this connection we have studied

the holographic superconductors. However, this leaves a possible scope to further apply this duality to describe several other interesting strongly interacting systems. Along with this, there remains the challenging issue of finding proper string theory embedding of these phenomenological models which may shed light on the properties of a class of strongly coupled field theories.

(iii) Owing to the non-relativistic nature of ordinary condensed matter systems, it seems reasonable to impose substantial priority in the study of gauge/gravity dualities in the framework of non-relativistic field theories. Further support in this regard may be provided since there are some serious experimental motivations behind these studies as mentioned earlier.

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Reprints

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Ehrenfest's scheme and thermodynamic geometry in Born-Infeld AdS black holes

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In this paper, we analyze the phase transition phenomena in Born-Infeld anti-de Sitter (BI AdS) black holes using Ehrenfest's scheme of standard thermodynamics. The critical points are marked by the divergences in the heat capacity. In order to investigate the nature of the phase transition, we analytically check both Ehrenfest equations near the critical points. Our analysis reveals that this is indeed a second order phase transition. Finally, we analyze the nature of the phase transition using the state space geometry approach. This is found to be compatible with Ehrenfest's scheme.

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I. INTRODUCTION

Thermodynamics of black holes in anti-de Sitter (AdS) space has received renewed attention since the discovery of the phase transition phenomena in the Schwarzschild AdS background [1]. To date, several attempts have been made in order to describe the phase transition phenomena in black holes [2–11]. All these works are basically based on the fact that at the critical point(s) of phase transition the specific heat of the black hole diverges. Despite of all these attempts, the issue regarding the classification of the nature of the phase transition in black holes remains highly debatable and worthy of further investigations. At this stage it is worthwhile to mention that in usual thermodynamics it is a general practice to adopt Ehrenfest's scheme [12] in order to classify the phase transition phenomena [13–17]. This is mainly due to two of its basic advantages, namely, (i) it is simple and elegant, and (ii) it provides a unique way to classify the nature of the phase transition in ordinary thermodynamic systems. Even if a phase transition is not truly a second order, we can determine the degree of its deviation by defining a new parameter called the Prigogine-Defay (PD) ratio (Π) [13,14,17]. Inspired by all these facts, we propose a possible way to overcome the long-standing problem regarding the classification of phase transition in black holes by incorporating the idea of Ehrenfest's scheme from standard thermodynamics. Since black holes in many respects behave as ordinary thermodynamic objects, the extension of Ehrenfest's scheme to black hole thermodynamics therefore seems to be quite natural. Such an attempt has been triggered very recently [18–23].

Constructing gravity theories in the presence of various higher derivative corrections to the usual Maxwell action has been a popular topic of research for the past several years [24–31]. Among these nonlinear theories of electrodynamics, it is the Born-Infeld theory that has earned renewed attention for the past few decades due to its several remarkable features. As a matter of fact, the Einstein-Born-Infeld theory admits various static black hole solutions that possess several significant qualitative features which are absent in ordinary Einstein-Maxwell gravity. These are given by the following: (i) Depending on the value of the electric charge (Q) and the Born-Infeld coupling parameter (b), it is observed that a meaningful black hole solution exists only for $bQ \geq 0.5$. On the other hand, for $bQ < 0.5$ the corresponding extremal limit does not exist. This eventually puts a restriction on the parameter space of BI AdS black holes [32]. This is indeed an interesting feature which is absent in the usual Einstein-Maxwell theory. (ii) Furthermore, one can note that for $bQ = 0.5$, which corresponds to the critical BI AdS case, there exists a new type of phase transition (HP3) which has identical thermodynamical features as observed during the phase transition phenomena in nonrotating BTZ black holes [32,33]. This is also an interesting observation that essentially leads to a remarkable thermodynamical analogy which does not hold in the corresponding Reissener-Nordström AdS (RN AdS) limit.

Motivated by all the above-mentioned features, in the present work we therefore aim to carry out a further investigation regarding the thermodynamic behavior of BI AdS black holes [32,34–37] in the framework of standard thermodynamics. We adopt Ehrenfest's scheme of usual thermodynamics in order to resolve a number of vexing issues regarding the phase transition phenomena in BI AdS black holes. The critical points correspond to an infinite discontinuity in the specific heat (C_Φ), which indicates the onset of a continuous higher order transition. At this point it is worthwhile to mention that an attempt to verify Ehrenfest's equations for charged RN AdS black holes was first initiated in Ref. [20]. There Ref. [20], based on numerical techniques, the authors computed both of Ehrenfest's equations close to the critical point(s). However, due the presence of infinite divergences in various thermodynamic entities (like heat capacity, etc.), as well as the lack of analytic techniques, at that time it was not possible to check Ehrenfest's equations exactly at the critical point(s). In order to address the above-mentioned

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issues, the present paper therefore aims to provide an analytic scheme in order to check Ehrenfest's equations exactly at the critical point(s). Moreover, the present analysis has been generalized taking the particular example of BI AdS black holes which is basically the nonlinear generalization of RN AdS black holes. Our analysis shows that it is indeed a second order phase transition.

Finally, we apply the widely explored state space geometry approach [38,39] to analyze the phase transition phenomena in BI AdS black holes [19,20,40–51]. Our analysis reveals that the scalar curvature (R) diverges exactly at the critical points where C_Φ diverges. This signifies the presence of a second order phase transition, thereby vindicating our earlier analysis based on Ehrenfest's scheme. It is also reassuring to note that in the appropriate limit ($b \longrightarrow \infty$, $Q \neq 0$), one recovers corresponding results for RN AdS black holes [20], where b is the Born-Infeld parameter and Q is the charge of the black hole.

Before we proceed further, let us mention about the organization of our paper. In Sec. II, we discuss thermodynamics of the BI black holes in AdS space. Using Ehrenfest's scheme, the nature of the phase transition is discussed in Sec. III. In Sec. IV, we analyze the phase transition using the (thermodynamic) state space geometry approach. Finally we draw our conclusion in Sec. V.

II. THERMODYNAMIC VARIABLES OF THE BI ADS BLACK HOLES

In order to obtain a finite total energy for the field around a pointlike charge, Born and Infeld proposed a nonlinear electrodynamics [52] in 1934. The Born-Infeld black hole solution with or without a cosmological constant is a nonlinear extension of the Reissner-Nordström black hole solution. In this paper, we will be concerned with the Born-Infeld solution in (3 + 1)-dimensional AdS space-time (with a negative cosmological constant $\Lambda = -3/l^2$) which is given by [32]

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2, \tag{1}$$

where we have taken the gravitational constant $G = 1$. Here the metric coefficient $f(r)$ is given by

$$f(r) = 1 - \frac{2M}{r} + r^2 + \frac{2b^2r^2}{3}\left(1 - \sqrt{1 + \frac{Q^2}{b^2r^4}}\right) + \frac{4Q^2}{3r^2}\mathcal{F}\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, \frac{-Q^2}{b^2r^4}\right), \tag{2}$$

where $\mathcal{F}(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, \frac{-Q^2}{b^2r^4})$ is the hypergeometric function [53] and b is the Born-Infeld parameter. In the limit $b \longrightarrow \infty$, $Q \neq 0$, one obtains the corresponding solution for the RN AdS space-time. At this stage it is reassuring to note that the rest of the analysis has been carried out keeping all terms in the hypergeometric series $\mathcal{F}(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, \frac{-Q^2}{b^2r^4})$.

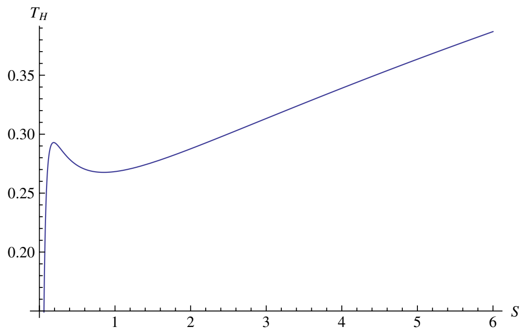

FIG. 1 (color online). Temperature (T) plot for BI AdS black hole with respect to entropy (S) for fixed $Q = 0.13$ and $b = 10$.

The ADM mass of the black hole is defined by $f(r_+) = 0$, which yields

$$M(r_+, Q, b) = \frac{r_+}{2} + \frac{r_+^3}{2} + \frac{b^2r_+^3}{3}\left(1 - \sqrt{1 + \frac{Q^2}{b^2r_+^4}}\right) + \frac{2Q^2}{3r_+}\mathcal{F}\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, \frac{-Q^2}{b^2r^4}\right), \tag{3}$$

where r_+ is the radius of the outer horizon.

Using (2), the Hawking temperature for the BI AdS black holes may be obtained as[1]

$$T = \frac{1}{4\pi}\left(\frac{df(r)}{dr}\right)_{r_+} = \frac{1}{4\pi}\left[\frac{1}{r_+} + 3r_+ + 2b^2r_+\left(1 - \sqrt{1 + \frac{Q^2}{b^2r_+^4}}\right)\right]. \tag{4}$$

From the first law of black hole thermodynamics we get $dM = TdS + \Phi dQ$. Using this we can obtain the entropy of the black hole as

$$S = \int_0^{r_+} \frac{1}{T}\left(\frac{\partial M}{\partial r}\right)_Q dr = \pi r_+^2 \tag{5}$$

Substituting (5) in (4) we can rewrite the Hawking temperature as [32]

$$T = \frac{1}{4\pi}\left[\sqrt{\frac{\pi}{S}} + 3\sqrt{\frac{S}{\pi}} + \frac{2b^2\sqrt{S}}{\sqrt{\pi}}\left(1 - \sqrt{1 + \frac{Q^2\pi^2}{b^2S^2}}\right)\right]. \tag{6}$$

We can see from Fig. 1 that there is a "hump" and a "dip" in the $T - S$ graph. Another interesting thing about the graph is that it is continuous in S. This rules out the possibility of first order phase transition. In order to check whether there is a possibility of higher order phase

[1] For details regarding the properties of hypergeometric function, see Ref. [53].

transition, we compute the specific heat at constant potential (C_Φ) [which is analog of the specific heat at constant pressure (C_P) in usual thermodynamics].

The electrostatic potential difference between the black hole horizon and the infinity is defined by Ref. [35] as

$$\Phi = \frac{Q}{r_+}\mathcal{F}\left(\frac{1}{4},\frac{1}{2},\frac{5}{4},\frac{-Q^2}{b^2 r_+^4}\right). \tag{7}$$

Using Ref. [5], we may further express Φ as

$$\Phi = \frac{Q\sqrt{\pi}}{\sqrt{S}}\mathcal{F}\left(\frac{1}{4},\frac{1}{2},\frac{5}{4},\frac{-Q^2\pi^2}{b^2 S^2}\right). \tag{8}$$

From the thermodynamical relation $T = T(S, Q)$, we find

$$\left(\frac{\partial T}{\partial S}\right)_\Phi = \left(\frac{\partial T}{\partial S}\right)_Q - \left(\frac{\partial T}{\partial Q}\right)_S\left(\frac{\partial \Phi}{\partial S}\right)_Q\left(\frac{\partial Q}{\partial \Phi}\right)_S, \tag{9}$$

where we have used the thermodynamic identity

$$\left(\frac{\partial Q}{\partial S}\right)_\Phi\left(\frac{\partial S}{\partial \Phi}\right)_Q\left(\frac{\partial \Phi}{\partial Q}\right)_S = -1. \tag{10}$$

Finally, using Eqs. (6), (8), and (9), the heat capacity C_Φ may be expressed as

$$C_\Phi = T\left(\frac{\partial S}{\partial T}\right)_\Phi = \frac{\mathcal{N}(Q, b, S)}{\mathcal{D}(Q, b, S)}, \tag{11}$$

where

$$\begin{aligned}\mathcal{N}(Q,b,S) &= -2S\left\{\pi + \left(3 - 2b^2\left(-1+\sqrt{1+\frac{Q^2\pi^2}{b^2S^2}}\right)\right)S\right\}\\ &\quad\times\left\{b^2S^2 + (\pi^2Q^2 + b^2S^2)\mathcal{F}\left(\frac{3}{4},1,\frac{5}{4},\frac{-Q^2\pi^2}{b^2S^2}\right)\right\}\end{aligned} \tag{12}$$

and

$$\begin{aligned}\mathcal{D}(Q,b,S) &= b^2S^2\left\{\pi + \left(-3 + 2b^2\left(-1+\sqrt{1+\frac{Q^2\pi^2}{b^2S^2}}\right)\right)S\right\}\\ &\quad + 2b^2S(-\pi Q + bS)(\pi Q + bS)\mathcal{F}\left(\frac{1}{4},\frac{1}{2},\frac{5}{4},\frac{-Q^2\pi^2}{b^2S^2}\right)\\ &\quad + (\pi - 3S - 2b^2S)(\pi^2Q^2 + b^2S^2)\mathcal{F}\left(\frac{3}{4},1,\frac{5}{4},\frac{-Q^2\pi^2}{b^2S^2}\right).\end{aligned} \tag{13}$$

Before going into the algebraic details, let us first plot $C_\Phi - S$ graphs (Figs. 2 and 3) and make a qualitative analysis of the plots.

From these figures it is evident that the heat capacity (C_Φ) suffers discontinuities exactly at two points (S_1 and S_2), which correspond to the critical points for the phase transition phenomena in BI AdS black holes. A similar conclusion also follows from the $T - S$ plot (Fig. 1), where the "hump" corresponds to S_1 and the "dip" corresponds to S_2.

The graph of $C_\Phi - S$ shows that there are three phases of the black hole: phase I ($0 < S < S_1$), phase II ($S_1 < S < S_2$), and phase III ($S > S_2$). Since the higher mass black hole possess larger entropy/horizon radius, at S_1 we therefore encounter a phase transition from a smaller mass black hole (phase I) to an intermediate (higher mass) black hole (phase II). On the other hand, S_2 corresponds to the critical point for the phase transition from the intermediate black hole (phase II) to a larger mass black hole (phase III). Finally, from Figs. 2 and 3 we also note that the heat capacity (C_Φ) is positive for phase I and phase III, whereas it is negative for phase II. Therefore, phase I and phase III correspond to the thermodynamically stable phases ($C_\Phi > 0$), whereas phase II stands for a thermodynamically unstable phase ($C_\Phi < 0$).

From the above discussions it is evident that the phase transitions we encounter in BI AdS black holes are indeed

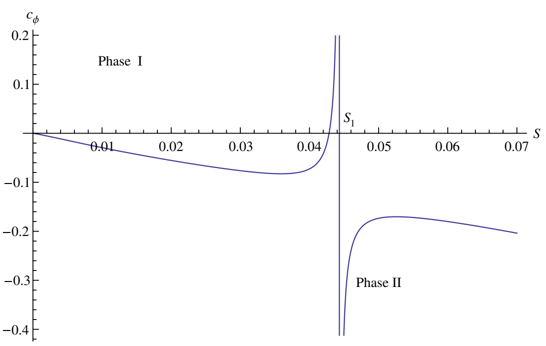

FIG. 2 (color online). Plot of specific heat (Plot of specific heat ($C\Phi$) against entropy (S) at the first critical point (S_1) for fixed $Q = 0.13$ and $b = 10$.

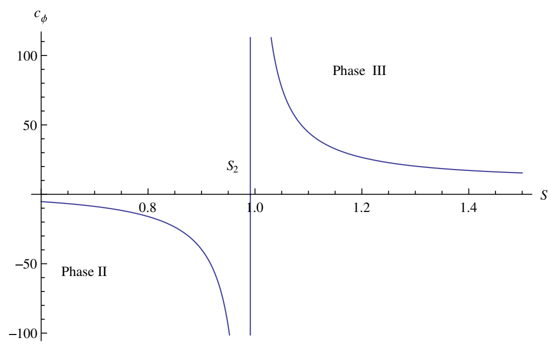

FIG. 3 (color online). Plot of specific heat ($C\Phi$) against entropy (S) at the second critical point (S_2) for fixed $Q = 0.13$ and $b = 10$.

continuous higher order. In order to address it more specifically (i.e., whether it is a second order or any higher order transition), we adopt a specific scheme known as Ehrenfest's scheme in standard thermodynamics.

III. STUDY OF PHASE TRANSITION USING EHRENFEST'S SCHEME

Discontinuity in the heat capacity does not always imply a second order transition, rather it suggests a continuous higher order transition in general. Ehrenfest's equations play an important role in order to determine the nature of such higher order transitions for various conventional thermodynamical systems [13–17]. This scheme can be applied in a simple and elegant way in standard thermodynamic systems. The nature of the corresponding phase transition can also be classified by applying this scheme. Moreover, even if a phase transition is not a genuine second order, we can determine the degree of its deviation by calculating the PD ratio [13,14,17]. Inspired by all these facts, we apply a similar technique to classify the phase transition phenomena in BI AdS black holes and check the validity of Ehrenfest's scheme for black holes.

In conventional thermodynamics, the first and second Ehrenfest equations [12] are given by

$$\left(\frac{\partial P}{\partial T}\right)_S = \frac{1}{VT}\frac{C_{P_2}-C_{P_1}}{\beta_2-\beta_1} = \frac{\Delta C_P}{VT\Delta\beta}, \tag{14}$$

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{\beta_2-\beta_1}{\kappa_2-\kappa_1} = \frac{\Delta\beta}{\Delta\kappa}. \tag{15}$$

In case of black hole thermodynamics, pressure (P) is replaced by the negative of the electrostatic potential difference ($-\Phi$), and volume (V) is replaced by charge (Q) of the black hole.

Thus, for the black hole thermodynamics, the two Ehrenfest equations become [18]

$$-\left(\frac{\partial \Phi}{\partial T}\right)_S = \frac{1}{QT}\frac{C_{\Phi_2}-C_{\Phi_1}}{\beta_2-\beta_1} = \frac{\Delta C_\Phi}{QT\Delta\beta}, \tag{16}$$

$$-\left(\frac{\partial \Phi}{\partial T}\right)_Q = \frac{\beta_2-\beta_1}{\kappa_2-\kappa_1} = \frac{\Delta\beta}{\Delta\kappa}, \tag{17}$$

respectively. Here, the subscripts 1 and 2 denote two distinct phases of the system. In addition, β is the *volume expansion coefficient* and κ is the *isothermal compressibility* of the system which are defined as

$$\beta = \frac{1}{Q}\left(\frac{\partial Q}{\partial T}\right)_\Phi, \tag{18}$$

$$\kappa = \frac{1}{Q}\left(\frac{\partial Q}{\partial \Phi}\right)_T. \tag{19}$$

Using Refs. [6,8] and considering the thermodynamic relation,

$$\left(\frac{\partial Q}{\partial T}\right)_\Phi = -\left(\frac{\partial \Phi}{\partial S}\right)_Q\left(\frac{\partial Q}{\partial \Phi}\right)_S\left(\frac{\partial S}{\partial T}\right)_\Phi,$$

we can show that

$$\beta = \frac{-8b^2\pi^{\frac{3}{2}}S^{\frac{5}{2}}}{\mathcal{D}(Q,b,S)}, \tag{20}$$

where the denominator was identified earlier [Eq. (13)].

In order to calculate κ, we make use of the thermodynamic identity,

$$\left(\frac{\partial Q}{\partial \Phi}\right)_T\left(\frac{\partial \Phi}{\partial T}\right)_Q\left(\frac{\partial T}{\partial Q}\right)_\Phi = -1. \tag{21}$$

Using (10) we obtain from (21),

$$\left(\frac{\partial Q}{\partial \Phi}\right)_T = \frac{(\frac{\partial T}{\partial S})_Q(\frac{\partial Q}{\partial \Phi})_S}{(\frac{\partial T}{\partial S})_\Phi}. \tag{22}$$

Using (6), (8), (9), and (22) we finally obtain

$$\kappa = \frac{\Psi(Q,b,S)}{\mathcal{D}(Q,b,S)}, \tag{23}$$

where

$$\Psi(Q,b,S) = \frac{-2b^2S^{\frac{3}{2}}}{Q\pi^{\frac{1}{2}}}\Bigg[2\pi^2Q^2 - \pi S\sqrt{1+\frac{Q^2\pi^2}{b^2S^2}} + \left(2b^2\left(-1+\sqrt{1+\frac{Q^2\pi^2}{b^2S^2}}\right) + 3\sqrt{1+\frac{Q^2\pi^2}{b^2S^2}}\right)\Bigg] \tag{24}$$

and the denominator $\mathcal{D}(Q,b,S)$ is given by (13).

Note that the denominators of both β and κ are indeed identical with that of C_Φ. This is manifested in the fact that both β and κ diverge exactly at the point(s) where C_Φ diverges (Figs. 4–7).

Being familiar with the two Ehrenfest equations and knowing β and κ, we can now determine the order of the phase transition. In order to do that we shall analytically check the validity of the two Ehrenfest equations at the points of discontinuity S_i $(i=1,2)$. Furthermore, it should be noted that at the points of divergence, we denote the critical values for the temperature (T) and charge (Q) as T_i and Q_i, respectively.

Let us now calculate the lhs of the first Ehrenfest Eq. (16), which may be written as

$$-\left[\left(\frac{\partial \Phi}{\partial T}\right)_S\right]_{S=S_i} = -\left[\left(\frac{\partial \Phi}{\partial Q}\right)_S\right]_{S=S_i}\left[\left(\frac{\partial Q}{\partial T}\right)_S\right]_{S=S_i}. \tag{25}$$

Using (6) and (8) we further obtain

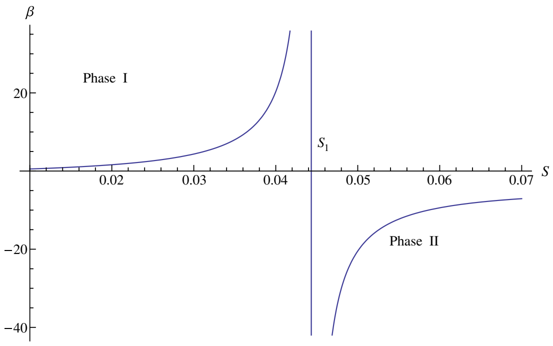

FIG. 4 (color online). Plot of volume expansion coefficient (β) against entropy (S) at the first critical point (S_1) for fixed $Q = 0.13$ and $b = 10$.

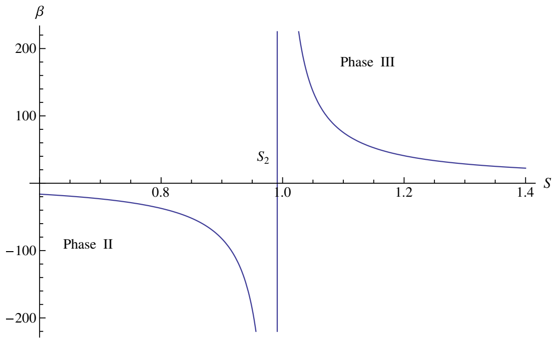

FIG. 5 (color online). Plot of volume expansion coefficient (β) against entropy (S) at the second critical point (S_2) for fixed $Q = 0.13$ and $b = 10$.

$$-\left[\left(\frac{\partial\Phi}{\partial T}\right)_S\right]_{S=S_i} = \frac{S_i}{Q_i}\left[1 + \left(1 + \frac{Q_i^2\pi^2}{b^2S_i^2}\right)\mathcal{F}\left(\frac{3}{4}, 1, \frac{5}{4}, \frac{-Q_i^2\pi^2}{b^2S_i^2}\right)\right]. \quad (26)$$

In order to calculate the rhs of the first Ehrenfest Eq. (16), we adopt the following procedure. From (11) and (18) we find

$$Q_i\beta = \left[\left(\frac{\partial Q}{\partial T}\right)_\Phi\right]_{S=S_i} = \left[\left(\frac{\partial Q}{\partial S}\right)_\Phi\right]_{S=S_i}\left(\frac{C_\Phi}{T_i}\right). \quad (27)$$

Therefore the rhs of (16) becomes

$$\frac{\Delta C_\Phi}{T_iQ_i\Delta\beta} = \left[\left(\frac{\partial S}{\partial Q}\right)_\Phi\right]_{S=S_i}. \quad (28)$$

Using (8) we can further write,

$$\frac{\Delta C_\Phi}{T_iQ_i\Delta\beta} = \frac{S_i}{Q_i}\left[1 + \left(1 + \frac{Q_i^2\pi^2}{b^2S_i^2}\right)\mathcal{F}\left(\frac{3}{4}, 1, \frac{5}{4}, \frac{-Q_i^2\pi^2}{b^2S_i^2}\right)\right]. \quad (29)$$

From (26) and (29) it is evident that both the lhs and rhs of the first Ehrenfest equation are indeed in good agreement at the critical points S_i. As a matter of fact, the divergence of C_Φ in the numerator is effectively canceled out by the diverging nature of β appearing at the denominator, which ultimately yields a finite value for the rhs of (16).

In order to calculate the lhs of the second Ehrenfest equation, we use the thermodynamic relation,

$$T = T(S, \Phi),$$

which leads to

$$\left(\frac{\partial T}{\partial\Phi}\right)_Q = \left(\frac{\partial T}{\partial S}\right)_\Phi\left(\frac{\partial S}{\partial\Phi}\right)_Q + \left(\frac{\partial T}{\partial\Phi}\right)_S. \quad (30)$$

Since C_Φ diverges at the critical points (S_i), it is evident from (11) that $[(\frac{\partial T}{\partial S})_\Phi]_{S=S_i} = 0$. Also, from (8) we find that $(\frac{\partial S}{\partial\Phi})_Q$ has a finite value at the critical points (S_i). Thus, from (30) and using (16) we may write,

$$-\left[\left(\frac{\partial\Phi}{\partial T}\right)_Q\right]_{S=S_i} = -\left[\left(\frac{\partial\Phi}{\partial T}\right)_S\right]_{S=S_i} = \frac{\Delta C_\Phi}{T_iQ_i\Delta\beta}. \quad (31)$$

From (19), at the critical points we can write,

$$\kappa Q_i = \left[\left(\frac{\partial Q}{\partial\Phi}\right)_T\right]_{S=S_i}. \quad (32)$$

Using (21) and (18) this can be further written as

$$\kappa Q_i = -\left[\left(\frac{\partial T}{\partial\Phi}\right)_Q\right]_{S=S_i} Q_i\beta. \quad (33)$$

Therefore, the rhs of (17) may be finally expressed as [22]

$$\frac{\Delta\beta}{\Delta\kappa} = -\left[\left(\frac{\partial\Phi}{\partial T}\right)_Q\right]_{S=S_i} \equiv -\left[\left(\frac{\partial\Phi}{\partial T}\right)_S\right]_{S=S_i} = \frac{S_i}{Q_i}\left[1 + \left(1 + \frac{Q_i^2\pi^2}{b^2S_i^2}\right)\mathcal{F}\left(\frac{3}{4}, 1, \frac{5}{4}, \frac{-Q_i^2\pi^2}{b^2S_i^2}\right)\right]. \quad (34)$$

This proves the validity of the second Ehrenfest equation at the critical points S_i. Finally, using (29) and (34), the PD ratio [13] may be obtained as

$$\Pi = \frac{\Delta C_\Phi\Delta\kappa}{T_iQ_i(\Delta\beta)^2} = 1, \quad (35)$$

which confirms the second order nature of the phase transition.

IV. STUDY OF PHASE TRANSITION USING STATE SPACE GEOMETRY

The validity of the two Ehrenfest equations near the critical points indeed suggests a second order nature of the phase transition. In order to analyze the phase transition phenomena from a different perspective, we adopt the

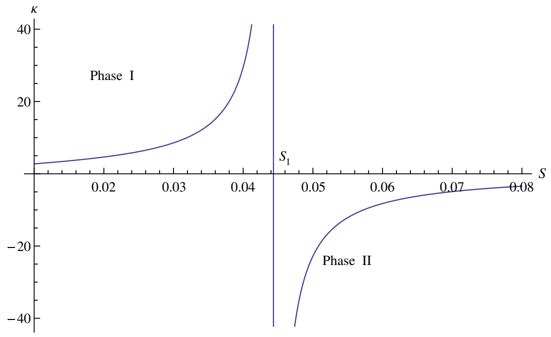

FIG. 6 (color online). Plot of isothermal compressibility (κ) against entropy (S) at the first critical point (S_1) for fixed $Q = 0.13$ and $b = 10$.

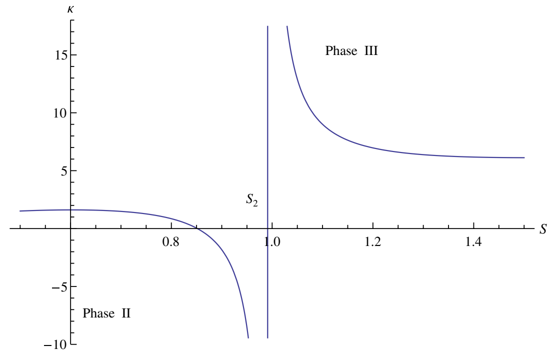

FIG. 7 (color online). Plot of isothermal compressibility (κ) against entropy (S) at the second critical point (S_2) for fixed $Q = 0.13$ and $b = 10$.

thermodynamic state space geometry approach [38,39], which has received renewed attention for the past two decades in the context of black hole thermodynamics [19,20,40–51]. This method provides an elegant way to analyze a second order phase transition near the critical point.

In the state space geometry approach, one aims to calculate the scalar curvature (R) [49] which suffers a discontinuity at the critical point for the second order phase transition.

In order to calculate the curvature scalar (R), we need to determine the Ruppeiner metric coefficients which may be defined as [38,39]

$$g^R_{ij} = -\frac{\partial^2 S(x^i)}{\partial x^i \partial x^j}, \tag{36}$$

where $x^i = x^i(M, Q)$, $i = 1, 2$ are the extensive variables of the system. From the computational point of view it is convenient to calculate the Weinhold metric coefficients [54]

$$g^W_{ij} = \frac{\partial^2 M(x^i)}{\partial x^i \partial x^j} \tag{37}$$

[where $x^i = x^i(S, Q)$, $i = 1, 2$] that are conformally connected to that of the Ruppeiner geometry through the following map [40,55]:

$$dS^2_R = \frac{dS^2_W}{T}. \tag{38}$$

In order to calculate g^R_{ij} we choose $x^1 = S$ and $x^2 = Q$. Finally, using (3) the Ruppeiner metric coefficients may be found as

$$g^R_{SS} = \frac{\frac{1}{2S}\left[-1 + \frac{3S}{\pi} + \frac{2b^2S}{\pi}\left(1 - \sqrt{1 + \frac{Q^2\pi^2}{b^2S^2}}\right) + \frac{4\pi Q^2}{S} - \frac{2\pi^3 Q^4}{b^2S^3}\right]}{\left[1 + \frac{3S}{\pi} + \frac{2b^2S}{\pi}\left(1 - \sqrt{1 + \frac{Q^2\pi^2}{b^2S^2}}\right)\right]} \tag{39}$$

$$g^R_{SQ} = \frac{\left(\frac{-2\pi Q}{S} + \frac{\pi^3 Q^3}{b^2S^3}\right)}{\left[1 + \frac{3S}{\pi} + \frac{2b^2S}{\pi}\left(1 - \sqrt{1 + \frac{Q^2\pi^2}{b^2S^2}}\right)\right]} \tag{40}$$

and

$$g^R_{QQ} = \frac{\left(4\pi - \frac{6\pi^3 Q^2}{5b^2S^2}\right)}{\left[1 + \frac{3S}{\pi} + \frac{2b^2S}{\pi}\left(1 - \sqrt{1 + \frac{Q^2\pi^2}{b^2S^2}}\right)\right]}. \tag{41}$$

Using these metric coefficients, the curvature scalar may be computed as

$$R = \frac{\wp(S, Q)}{\Re(S, Q)}. \tag{42}$$

The numerator $\wp(S, Q)$ is too cumbersome, which prevents us from presenting its detailed expression for the present work. However, the denominator $\Re(S, Q)$ may be expressed as

$$\Re = \sqrt{1 + \frac{\pi^2 Q^2}{b^2S^2}}\left[-\pi + \left(-3 + 2b^2\left(-1 + \sqrt{1 + \frac{\pi^2Q^2}{b^2S^2}}\right)\right)S\right]\left(\pi^2Q^2 + b^2S^2\right)\left(-\pi^6Q^6 + 12b^2\pi^4Q^4S^2 - 3b^2\pi^3Q^2S^3 + b^2\pi^2Q^2\left(9 - 2b^2\left(7 + 3\sqrt{1 + \frac{\pi^2Q^2}{b^2S^2}}\right)\right)S^4 + 10b^4\pi S^5 + 10b^4\left(-3 + 2b^2\left(-1 + \sqrt{1 + \frac{\pi^2Q^2}{b^2S^2}}\right)\right)S^6\right)^3. \tag{43}$$

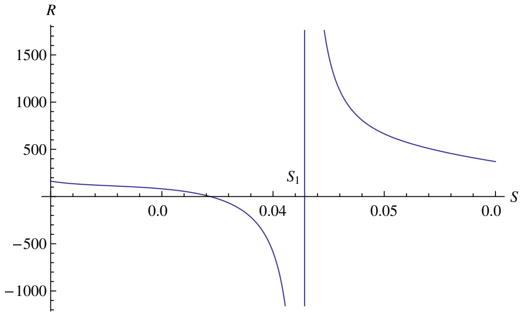

FIG. 8 (color online). Plot of scalar curvature (R) against entropy (S) at the first point of divergence (S_1) (antisymmetric divergence) for fixed $Q = 0.13$ and $b = 10$.

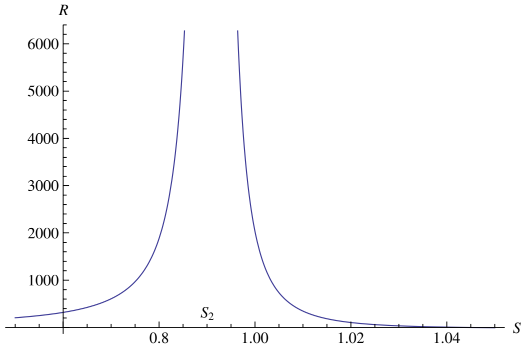

FIG. 9 (color online). Plot of scalar curvature (R) against entropy (S) at the second point of divergence ($S2$) (symmetric divergence) for fixed $Q = 0.13$ and $b = 10$.

Before we conclude this section, let us note few interesting points at this stage. First of all, from Figs. 8 and 9 we observe that the scalar curvature (R) diverges exactly at the points where the specific heat (C_Φ) diverges (also see Figs. 2 and 3). This is an expected result, since according to Ruppeiner any divergence in R implies a corresponding divergence in the heat capacity C_Φ (which is thermodynamic analog of C_P) that essentially leads to a change in stability [49]. On the other hand, no divergence in R could be observed at the Davis critical point [56], which is marked by the divergence in C_Q (which is the thermodynamic analog of C_V). Similar features have also been observed earlier [19,20].

V. CONCLUSIONS

In this paper, based on a standard thermodynamic approach, we systematically analyze the phase transition phenomena in BI AdS black holes. Our results are valid for all orders in the BI parameter (b). The continuous nature of the $T - S$ plot essentially rules out the possibility of any first order transition. On the other hand, the discontinuity of the heat capacity (C_Φ) indicates the onset of a continuous higher order transition. In order to address this issue further, we provide a detailed analysis of the phase transition phenomena using Ehrenfest's scheme of standard thermodynamics [12] which uniquely determines the second order nature of the phase transition. At this stage it is reassuring to note that the first application of Ehrenfest's scheme in order to determine the nature of phase transition in charged RN AdS black holes had been commenced in Ref. [20]. There, the analysis had been carried out *numerically* in order to check the validity of the Ehrenfest equations close to the critical point(s). Frankly speaking, the analysis presented in Ref. [20] was actually in an underdeveloped stage. This is mainly due to the fact that at that time no such *analytic* scheme was available. As a result, at that stage of analysis it was not possible to check the validity of the Ehrenfest equations exactly at the critical point(s) due to the occurrence of infinite divergences of various thermodynamic entities at the (phase) transition point(s).

In order to address the above-mentioned issue, in the present paper we provide an analytical scheme to check the validity of the Ehrenfest equations exactly at the critical point(s). Furthermore, we carry out the entire analysis taking the particular example of BI AdS black holes which is basically the nonlinear generalization of RN AdS black holes. Therefore, our results are quite general and hence they are valid for a wider class of charged black holes in the usual Einstein gravity.

Also, we analyze the phase transition phenomena using the state space geometry approach. Our analysis shows that the scalar curvature (R) suffers discontinuities exactly at the (critical) points where the heat capacity (C_Φ) diverges. This further indicates the second order nature of the phase transition. Therefore, from our analysis it is clear that both Ehrenfest's scheme and the state space geometry approach essentially lead to an identical conclusion. This also establishes their compatibility while studying phase transitions in black holes.

Finally, we remark that the curvature scalar, which behaves in a very suggestive way for conventional systems, displays similar properties for black holes. Specifically, a diverging curvature that signals the occurrence of a second order phase transition in usual systems retains this characteristic for black holes. Our analysis reveals a direct connection between Ehrenfest's scheme and thermodynamic state space geometry.

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Research Article

Critical Phenomena in Higher Curvature Charged AdS Black Holes

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In this paper, we have studied the critical phenomena in higher curvature charged AdS black holes. We have considered Lovelock-Born-Infeld-AdS black hole as an example. The thermodynamics of the black hole have been studied which reveals the onset of a higher-order phase transition in the black hole in the canonical ensemble (fixed charge ensemble) framework. We have analytically derived the critical exponents associated with these thermodynamic quantities. We find that our results fit well with the thermodynamic scaling laws and consistent with the mean field theory approximation. The suggestive values of the other two critical exponents associated with the correlation function and correlation length on the critical surface have been derived.

1. Introduction

Constructing gravity theories in higher dimensions (i.e., greater than four) has been an interesting topic of research for the past several decades. One of the reasons for this is that these theories provide a framework for unifying gravity with other interactions. Since string theory is an important candidate for such unified theory, it is necessary to consider higher-dimensional space-time for its consistency. In fact, the study of string theory in higher dimensions is one of the most important and challenging sectors in high energy physics. String theory requires the inclusion of gravity in order to describe some of its fundamental properties. Other theories like brane theory are theory of supergravity are also studied in higher dimensions. All these above mentioned facts underscored the necessity for considering gravity theories in higher dimensions [1]. The effect of string theory on gravity may be understood by considering a low energy effective action that describes the classical gravity [2]. This effective action must include combinations of the higher curvature terms and found to be ghost-free [3]. In an attempt to obtain the most general tensor that satisfies the properties of Einstein's tensor in higher dimensions, Lovelock proposed an effective action that contains higher curvature terms [4]. The field equations derived from this action consist of only second derivatives of the metric and hence are free of ghosts [5]. In fact, these theories are the most general second-order gravity theories in higher dimensions.

Among various higher dimensional theories of gravity, it is the seven-dimensional gravity that has earned repeated interests over the past two decades because of the interesting physics associated with them. For example, addition of certain topological terms in the usual Einstein-Hilbert action produces new type of gravity in seven-dimensions. For a particular choice of the coupling constants it is observed that these terms exist in seven dimensional gauged supergravity [6]. Apart from these aspects of the supergravity theories in 7 dimensions, there are several other important features associated with gravity theories in these particular dimensions. For example, there exists an octonionic instanton solution to the seven-dimensional Yang-Mills theory having an extension to a solitonic solution in low energy heterotic string theory [7]. Emergence of U-duality in seven dimensions involves some nontrivial phenomena which are interpreted in M-theory [8]. Also, study of black holes in 7 dimensions is important in the context of the AdS/CFT correspondence [9–11].

There has been much progress in the study of various properties of the black holes both in four as well as in higher dimensions (greater than four) over the past few decades. Black holes in higher dimensions possess interesting properties which may be absent in four dimensions [12–16] (for an excellent review on black holes in higher dimensions see [17]). Among several intriguing properties of black holes we will be mainly concerned with the thermodynamics of these objects. This motivation primarily arises from the fact that the study of black hole thermodynamics in higher dimensions may provide us with information about the nature of quantum gravity. The limits of validity of the laws of black hole mechanics [18] may address some interpretations in this regard. Various quantum effects can be imposed in order to check the validity of these laws. For example, if we want to study the effect of backreaction of quantum field energy on the laws on black hole mechanics, we are required to consider higher curvature interactions [19]. From this point of view it is very much natural to study the effect of higher curvature terms on the thermodynamic properties of black holes. In fact, the higher-dimensional Lovelock theory sets a nice platform to study this effect. Moreover, the higher curvature gravity theories in higher dimensions are free from the complications that arise from the higher derivative theories. All these facts motivate us to study the higher curvature gravity theories in higher dimensions.

The study of thermodynamic properties of black holes in anti-de Sitter (AdS) space-time has got renewed attention due to the discovery of phase transition in Schwarzschild-AdS black holes [20]. Since then a wide variety of researches have been commenced in order to study the phase transition in black holes [15, 16, 21–33]. Nowadays, the study of thermodynamics of black holes in AdS space-time is very much important in the context of AdS/CFT duality. The thermodynamics of AdS black holes may provide important information about the underlying phase structure and thermodynamic properties of CFTs [34]. The study of thermodynamic phenomena in black holes requires an analogy between the variables in ordinary thermodynamics and those in black hole mechanics. Recently, Banerjee et al. have developed a method [28, 29], based on Ehrenfest's scheme [35] of standard thermodynamics, to study the phase transition phenomena in black holes. In this approach one can actually determine the order of phase transition once the relevant thermodynamic variables are identified for the black holes. This method has been successfully applied in the four dimensional black holes in AdS space-time [29–32] as well as in higher dimensional AdS black holes [15].

The behavior of the thermodynamic variables near the critical points can be studied by means of a set of indices known as static critical exponents [36, 37]. These exponents are to a large extent universal and independent of the spatial dimensionality of the system and obey thermodynamic scaling laws [36, 37]. The critical phenomena have been studied extensively in familiar physical systems like the Ising model (two and three dimensions), magnetic systems, elementary particles, hydrodynamic systems, and so forth. An attempt to study the critical phenomena in black holes was commenced in the last twenty years [38–52]. Despite all these attempts, a systematic study of critical phenomena in black holes was still lacking. This problem has been circumvented very recently [16, 53]. In these works the critical phenomena have been studied in (3 + 1) as well as higher dimensional AdS black holes. Also, the critical exponents of the black holes have been determined by explicit analytic calculations.

All the researches mentioned above were confined to Einstein gravity. It would then be interesting to study gravity theories in which action involves higher curvature terms. Among the higher curvature black holes, Gauss-Bonnet and Lovelock black holes will be suitable candidates to study. The thermodynamic properties and phase transitions have been studied in the Gauss-Bonnet AdS (GB-AdS) black holes [22, 54–58]. Also, the critical phenomena in the GB-AdS black holes were studied [59]. On the other hand, Lovelock gravity coupled to the Maxwell field was investigated in [60, 61]. The thermodynamic properties of the third order Lovelock-Born-Infeld-AdS (LBI-AdS) black holes in seven space-time dimensions were studied in [60, 62, 63]. But the study of phase transition and critical phenomena have not yet been done in these black holes.

In this paper we have studied the critical phenomena in the third-order LBI-AdS black holes in seven dimensions. We have also given a qualitative discussion about the possibility of higher order phase transition in this type of black holes. We have determined the static critical exponents for these black holes and showed that these exponents obey the static scaling laws. We have also checked the static scaling hypothesis and calculated the scaling parameters. We observe that these critical exponents take the mean field values. From our study of the critical phenomena we may infer that the third order LBI-AdS black holes yield results consistent with the mean field theory approximation. Despite having some distinct features in the higher curvature gravity theories, for example, the usual area law valid in Einstein gravity does not hold in these gravity theories, the critical exponents are found to be identical with those in the usual Einstein gravity. This result shows that the AdS black holes, studied so far, belong to the same universality class. We have also determined the critical exponents associated with the correlation function and correlation length. However, the values of these exponents are more suggestive than definitive. As a final remark, we have given a qualitative argument for the determination of these last two exponents.

The organization of the paper is as follows. In Section 2 we discuss the thermodynamical variables of the seven-dimensional third order LBI-AdS black holes. In Section 3 we analyze the phase transition and stability of these black holes. The critical exponents, scaling laws, and scaling hypothesis are discussed in Section 4. Finally, we have drawn our conclusions in Section 5.

2. Thermodynamic Variables of Higher Curvature Charged AdS Black Holes

The effective action in the Lovelock gravity in $(n+1)$ dimensions can be written as (here we have taken the gravitational constant $G = 1$) [4]

$$\mathcal{I} = \frac{1}{16\pi}\int d^{n+1}x\sqrt{-g}\sum_{i=0}^{[(n+1)/2]}\alpha_i\mathcal{L}_i, \quad (1)$$

where α_i is an arbitrary constant and $\mathcal{L}_i$ is the *Euler density* of a $2i$-dimensional manifold. In $(n+1)$ dimensions all terms for which $i > [(n+1)/2]$ are equal to zero, the term $i = (n+1)/2$ is a topological term, and terms for which $i < [(n+1)/2]$ contribute to the field equations. Since we are studying third-order Lovelock gravity in the presence of Born-Infeld nonlinear electrodynamics [64], the effective action of (1) may be written as

$$\begin{aligned}\mathcal{I} &= \frac{1}{16\pi}\int d^{n+1}x\sqrt{-g}\,\left(\alpha_0\mathcal{L}_0 + \alpha_1\mathcal{L}_1 + \alpha_2\mathcal{L}_2 + \alpha_3\mathcal{L}_3 + L(F)\right) \\ &= \frac{1}{16\pi}\int d^{n+1}x\sqrt{-g}\,\left(-2\Lambda + \mathcal{R} + \alpha_2\mathcal{L}_2 + \alpha_3\mathcal{L}_3 + L(F)\right),\end{aligned} \tag{2}$$

where Λ is the cosmological constant given by $-n(n-1)/2l^2$, l being the AdS length, α_2 and α_3 are the second- and third-order Lovelock coefficients, $\mathcal{L}_1 = \mathcal{R}$ is the usual Einstein-Hilbert Lagrangian, $\mathcal{L}_2 = R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + \mathcal{R}^2$ is the Gauss-Bonnet Lagrangian,

$$\begin{aligned}\mathcal{L}_3 &= 2R^{\mu\nu\sigma\kappa}R_{\sigma\kappa\rho\tau}R^{\rho\tau}_{\mu\nu} + 8R^{\mu\nu}_{\sigma\rho}R^{\sigma\kappa}_{\nu\tau}R^{\rho\tau}_{\mu\kappa} \\ &\quad + 24R^{\mu\nu\sigma\kappa}R_{\sigma\kappa\nu\rho}R^{\rho}_{\mu} + 3\mathcal{R}R^{\mu\nu\sigma\kappa}R_{\sigma\kappa\mu\nu} \\ &\quad + 24R^{\mu\nu\sigma\kappa}R_{\sigma\mu}R_{\kappa\nu} + 16R^{\mu\nu}R_{\nu\sigma}R^{\sigma}_{\mu} - 12\mathcal{R}R^{\mu\nu}R_{\mu\nu} + \mathcal{R}^3\end{aligned} \tag{3}$$

is the third-order Lovelock Lagrangian, and $L(F)$ is the Born-Infeld Lagrangian given by

$$L(F) = 4b^2\left(1 - \sqrt{1 + \frac{F^2}{2b^2}}\right). \tag{4}$$

In (4), $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $F^2 = F_{\mu\nu}F^{\mu\nu}$, and b is the Born-Infeld parameter. In the limit $b \to \infty$ we recover the standard Maxwell form $L(F) = -F^2$.

The solution of the third order Lovelock-Born-Infeld anti de-Sitter black hole (LBI-AdS) in $(n+1)$-dimensions can be written as [63]

$$ds^2 = -f(r)\,dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2_{k,n-1}, \tag{5}$$

where

$$d\Omega^2_{k,n-1} = \begin{cases} d\theta_1^2 + \sum_{i=2}^{n-1}\prod_{j=1}^{i-1}\sin^2\theta_j d\theta_i^2 & k = 1 \\ d\theta_1^2 + \sinh^2\theta_1 d\theta_2^2 + \sinh^2\theta_1\sum_{i=3}^{n-1}\prod_{j=2}^{i-1}\sin^2\theta_j d\theta_i^2 & k = -1 \\ \sum_{i=1}^{n-1} d\phi_i^2 & k = 0 \end{cases} \tag{6}$$

and k determines the structure of the black hole horizon ($k = +1$ (spherical), -1 (hyperbolic), 0 (planar)). At this point of discussion it must be mentioned that the Lagrangian of (2) is the most general Lagrangian in seven space-time dimensions that produce the second-order field equations [60]. Thus, we will restrict ourselves in the seven space-time dimensions.

The equation of motion for the electromagnetic field for this $(6+1)$-dimensional space-time can be obtained by varying the action (2) with respect to the gauge field A_μ. This results in the following [63]:

$$\partial_\mu\left(\frac{\sqrt{-g}F^{\mu\nu}}{\sqrt{1 + F^2/2b^2}}\right) = 0, \tag{7}$$

which has a solution [63]

$$A_\mu = -\sqrt{\frac{5}{8}}\left(\frac{q}{r^4}\right)\mathcal{H}(\eta)\,\delta^0_\mu. \tag{8}$$

Here $\mathcal{H}(\eta)$ is the abbreviation of the hypergeometric function [65] given by

$$\mathcal{H}(\eta) = \mathcal{H}\left(\frac{1}{2}, \frac{2}{5}, \frac{7}{5}, -\eta\right). \tag{9}$$

In (8), $\eta = 10q^2/b^2r^{10}$ and q is a constant of integration which is related to the charge (Q) of the black hole. The charge (Q) of the black hole may be obtained by calculating the flux of the electric field at infinity [60, 63, 66]. Therefore,

$$\begin{aligned}Q &= -\int_{\mathcal{B}} d^{n-1}\omega\sqrt{\sigma}n_\mu\tau_\nu\left(\frac{F^{\mu\nu}}{\sqrt{1 + F^2/2b^2}}\right) \\ &= \frac{\mathcal{V}_{n-1}}{4\pi}\sqrt{\frac{(n-1)(n-2)}{2}}q \\ &= \frac{\sqrt{10}\pi^2 q}{4} \quad (\text{for } n = 6),\end{aligned} \tag{10}$$

where n_μ and τ_ν are the time-like and space-like unit normal vectors to the boundary $\mathcal{B}$, respectively, and σ is the determinant of the induced metric σ_{ij} on $\mathcal{B}$ having coordinates ω^i. It is to be noted that in deriving (10) we have only considered the F^{0r} component of $F^{\mu\nu}$.

The quantity $\mathcal{V}_{n-1}$ is the volume of the $(n-1)$ sphere and may be written as

$$\mathcal{V}_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{11}$$

The metric function $f(r)$ of (5) may be written as [63]

$$f(r) = k + \frac{r^2}{\alpha'}\left(1 - \chi(r)^{1/3}\right), \tag{12}$$

where

$$\chi(r) = 1 + \frac{3\alpha' m}{r^6} - \frac{2\alpha' b^2}{5}\left[1 - \sqrt{1+\eta} - \frac{\Lambda}{2b^2} + \frac{5\eta}{4}\mathcal{H}(\eta)\right]. \tag{13}$$

Here we have considered the special case $\alpha_3 = 2\alpha_2^2 = \alpha'^2/72$ [60, 63].

Since, in our study of the critical phenomena in the third-order LBI-AdS black holes, the thermodynamic quantities like "quasilocal energy" (M), Hawking temperature (T), entropy (S), and so forth will play important roles, we will now focus mainly on the derivations of these quantities.

The "quasilocal energy" M of asymptotically AdS black holes may be obtained by using the counterterm method which is indeed inspired by the AdS/CFT correspondence [67–69]. This is a well-known technique which removes the divergences in the action and conserved quantities of the associated space-time. These divergences appear when one tries to add surface terms to the action in order to make it well defined. The counterterm method was applied earlier for the computation of the conserved quantities associated with space-time, having finite boundaries, de-Sitter (dS) space-time and asymptotically flat space-time in the framework of Einstein gravity [70–76]. On the other hand, this method was applied in Lovelock gravity to compute the associated conserved quantities [60, 61, 63, 77, 78]. However, for the asymptotically AdS solutions of the third order Lovelock black holes the action may be written as [61, 63, 78]

$$\mathscr{A} = \mathscr{I} + \underbrace{\frac{1}{8\pi}\int_{\partial\mathscr{M}} d^n x\sqrt{|\gamma|}\left(\mathscr{I}_b^1 + \alpha_2\mathscr{I}_b^2 + \alpha_3\mathscr{I}_b^3\right)}_{\text{boundary terms}} + \underbrace{\frac{1}{8\pi}\int_{\partial\mathscr{M}} d^n x\sqrt{|\gamma|}\left(\frac{n-1}{L}\right)}_{\text{counter term}}, \tag{14}$$

where γ is the determinant of the induced metric γ_{ab} on the time-like boundary $\partial\mathscr{M}$ of the space-time manifold $\mathscr{M}$. The quantity L is a scale length factor given by

$$L = \frac{15\sqrt{\alpha'(1-\lambda)}}{5 + 9\alpha' - \lambda^2 - 4\lambda}, \tag{15}$$

where

$$\lambda = \left(1 - 3\alpha'\right)^{1/3}. \tag{16}$$

The boundary terms in (14) are chosen such that the action possesses well-defined variational principle, whereas, the counterterm makes the action and the associated conserved quantities finite.

The boundary terms appearing in (14) may be identified as [61, 63, 78]

$$\mathscr{I}_b^1 = K \tag{17}$$

$$\mathscr{I}_b^2 = 2\left(J - 2\overline{G}_{ab}^1 K^{ab}\right) \tag{18}$$

$$\mathscr{I}_b^3 = 3\Big(P - 2\overline{G}_{ab}^2 K^{ab} - 12\overline{R}_{ab}J^{ab} + 2\overline{\mathscr{R}}J - 4K\overline{R}_{abcd}K^{ac}K^{bd} - 8K\overline{R}_{abcd}K^{ac}K_e^b K^{ed}\Big), \tag{19}$$

where K is the trace of the extrinsic curvature K_{ab}. In (18) $\overline{G}_{ab}^1$ is the Einstein tensor for γ_{ab} in n-dimensions, and J is the trace of the following tensor:

$$J_{ab} = \frac{1}{3}\left(2KK_{ac}K_b^c + K_{cd}K^{cd}K_{ab} - 2K_{ac}K^{cd}K_{db} - K^2 K_{ab}\right). \tag{20}$$

In (19) $\overline{G}_{ab}^2$ is the second order Lovelock tensor for γ_{ab} in n-dimensions which is given by

$$\overline{G}_{ab}^2 = 2\left(\overline{R}_{acde}\overline{R}_b^{cde} - 2\overline{R}_{afbc}\overline{R}^{fc} - 2\overline{R}_{ac}\overline{R}_b^c + \overline{\mathscr{R}}\overline{R}_{ab}\right) - \mathscr{L}_2(\gamma)\gamma_{ab}, \tag{21}$$

whereas P is the trace of

$$\begin{aligned} P_{ab} = \frac{1}{5}\Big(\Big[&K^4 - 6K^2K^{cd}K_{cd} + 8KK_{cd}K_e^dK^{ec} \\ &- 6K_{cd}K^{de}K_{ef}K^{fc} + 3\left(K_{cd}K^{cd}\right)^2\Big]K_{ab} \\ &- \left(4K^3 - 12KK_{ed}K^{ed} + 8K_{de}K_f^eK^{fd}\right)K_{ac}K_b^c \\ &- 24KK_{ac}K^{cd}K_{de}K_b^e \\ &+ \left(12K^2 - 12K_{ef}K^{ef}\right)K_{ac}K^{cd}K_{db} \\ &+ 24K_{ac}K_{cd}K_{de}K^{ef}K_{fb}\Big). \end{aligned} \tag{22}$$

Using the method prescribed in [79], we obtain the divergence-free energy-momentum tensor as [61, 63, 78],

$$T^{ab} = \frac{1}{8\pi}\left[\left(K^{ab} - K\gamma^{ab}\right) + 2\alpha_2\left(3J^{ab} - J\gamma^{ab}\right) + 3\alpha_3\left(5P^{ab} - P\gamma^{ab}\right) + \frac{n-1}{L}\gamma^{ab}\right]. \tag{23}$$

The first three terms of (23) result from the variation of the boundary terms of (14) with respect to the induced metric γ^{ab}, whereas the last term is obtained by considering the variation of the counterterm of (14) with respect to γ^{ab}.

For any space-like surface $\mathscr{B}$ in $\partial\mathscr{M}$ which has the metric σ_{ij} we can write the boundary metric in the following form [61, 63, 78, 79]:

$$\gamma_{ab}dx^a dx^b = -N^2dt^2 + \sigma_{ij}\left(d\omega^i + V^i dt\right)\left(d\omega^j + V^j dt\right), \tag{24}$$

where ω^i are coordinates on $\mathscr{B}$ and N and V^i are the lapse function and the shift vector, respectively. For any Killing vector field ξ on the the space-like boundary $\mathscr{B}$ in $\partial\mathscr{M}$, we may write the conserved quantities associated with the energy momentum tensor (T^{ab}) as [61, 63, 78, 79]

$$\mathscr{Q}_\xi = \int_{\mathscr{B}} d^{n-1}\omega\sqrt{\sigma}n^a\xi^b T_{ab}, \tag{25}$$

where σ is the determinant of the metric σ_{ij} on $\mathscr{B}$, n^a is the time-like unit normal to $\mathscr{B}$, and $\xi^b(=\partial/\partial t)$ is the time-like Killing vector field. For the metric (5) we can write $n^a = (1/\sqrt{f(r)}, 0, 0, 0, \ldots)$, and $\xi^a = (1, 0, 0, \ldots)$, $K_{ab} = -\gamma_a^m\nabla_m\tau_b$, where $\tau^a = (0, f(r), 0, \ldots)$ is the space-like unit normal to the boundary.

With these values of n^a and ξ^a the only nonvanishing component of T_{ab} becomes T_{00}. Hence, $\mathscr{Q}_\xi$ corresponds to the "quasilocal energy" M of the black hole. Thus, from (25) the expression for the "quasilocal energy" of the black hole may be written as

$$M = \int_{\mathscr{B}} d^{n-1}\omega\sqrt{\sigma}n^0\xi^0 T_{00}. \tag{26}$$

Using (12) and (23), (26) can be computed as

$$\begin{aligned} M|_{n=6} &= \frac{\mathscr{V}_{n-1}}{16\pi}(n-1)\,m|_{n=6} \\ &= \frac{5\pi^2}{16}m, \end{aligned} \tag{27}$$

where the constant m is expressed as the real root of the equation

$$f(r = r_+) = 0. \tag{28}$$

Using (11) and substituting m from (28) we finally obtain from (27)

$$\begin{aligned} M = \frac{5\pi^2}{16}\Bigg[&\frac{k^3\alpha'^2}{3} + kr_+^4 + k^2\alpha' r_+^2 \\ &+ \frac{2b^2 r_+^6}{15}\left(1 - \sqrt{1+\eta_+} - \frac{\Lambda}{2b^2} + \frac{20Q^2}{b^2\pi^4 r_+^{10}}\mathscr{H}(\eta_+)\right)\Bigg]. \end{aligned} \tag{29}$$

The electrostatic potential difference between the black hole horizon and the infinity may be defined as [63]

$$\Phi = \sqrt{\frac{(n-1)}{2(n-2)}}\frac{q}{r_+^{n-2}}\mathscr{H}(\eta_+) = \frac{Q}{\pi^2 r_+^4}\mathscr{H}(\eta_+) \quad (\text{for } n = 6), \tag{30}$$

where

$$\eta_+ = \frac{16Q^2}{b^2\pi^4 r_+^{10}}. \tag{31}$$

It is to be noted that in obtaining (31) we have used (10).

The Hawking temperature for the third order LBI-AdS black hole is obtained by analytic continuation of the metric. If we set $t \to i\tau$, we obtain the Euclidean section of (12) which requires to be regular at the horizon (r_+). Thus we must identify $\tau \sim \tau + \beta$, where β ($= 2\pi/\kappa$, κ being the surface gravity of the black hole) is the inverse of the Hawking temperature. Therefore the Hawking temperature may be written as

$$\begin{aligned} &\beta^{-1} \\ &= T = \frac{1}{4\pi}\left(\frac{\partial f(r)}{\partial r}\right)_{r_+} \\ &= \frac{10kr_+^4 + 5k\alpha' r_+^2 + 2b^2 r_+^6\left(1 - \sqrt{1 + 16Q^2/b^2\pi^4 r_+^{10}}\right) - \Lambda r_+^6}{10\pi r_+\left(r_+^2 + k\alpha'\right)^2}. \end{aligned} \tag{32}$$

It is interesting to note that in the limit $\alpha' \to 0$, the corresponding expression for the Hawking temperature of the Born-Infeld AdS (BI-AdS) black hole can be recovered as [16]

$$T_{\text{BI-AdS}} = \frac{1}{4\pi}\left[\frac{4k}{r_+} + \frac{4b^2 r_+}{5}\left(1 - \sqrt{1 + \frac{Q^2}{b^2 r_+^4}}\right) - \frac{2\Lambda r_+}{5}\right]. \tag{33}$$

The entropy of the black hole may be calculated from the first law of black hole mechanics [80]. In fact, it has been found that the thermodynamic quantities (e.g., entropy, temperature, "quasilocal energy," etc.) of the LBI-AdS black holes satisfy the first law of black hole mechanics [60, 61, 63, 66, 77, 81]:

$$dM = TdS + \Phi dQ. \tag{34}$$

Using (34) the entropy of the black hole may be obtained as

$$\begin{aligned} S &= \int_0^{r_+}\frac{1}{T}\left(\frac{\partial M}{\partial r_+}\right)_Q dr_+ \\ &= \frac{\pi^3}{4}\left(r_+^5 + \frac{10k\alpha' r_+^3}{3} + 5k^2\alpha'^2 r_+\right), \end{aligned} \tag{35}$$

where we have used (29) and (32). At this point it is interesting to note that identical expression for the entropy was obtained earlier using somewhat different approach [82–86]. In this approach, in an arbitrary spatial dimension n, the expression for the Wald entropy for higher curvature black holes is written as

$$\begin{aligned} S &= \frac{1}{4}\sum_{j=1}^{[(n-2)/2]} j\alpha_j \int_{\mathscr{B}} d^{n-1}\omega\sqrt{|\sigma|}\mathscr{L}_{j-1}(\sigma) \\ &= \frac{1}{4}\int_{\mathscr{B}} d^{n-1}\omega\sqrt{|\sigma|} \\ &\quad\times\left(1 + 2\alpha_2\widetilde{\mathscr{R}} + 3\alpha_3\left(\widetilde{R}_{abcd}\widetilde{R}^{abcd} - 4\widetilde{R}_{ab}\widetilde{R}^{ab} + \widetilde{\mathscr{R}}^2\right)\right) \\ &= \frac{\mathscr{V}_{n-1}}{4}\left[r_+^4 + \frac{2(n-1)}{(n-3)}k\alpha' r_+^2 + \frac{(n-1)}{(n-5)}k^2\alpha'^2\right] r_+^{n-5}, \end{aligned} \tag{36}$$

where $\mathscr{L}_j(\sigma)$ is the jth order Lovelock Lagrangian of σ_{ij} and the *tilde* denotes the corresponding quantities for the induced metric σ_{ij}. If we put $n = 6$ in (36) we obtain the expression of (35) (the entropy of black holes both in the usual Einstein gravity and in higher curvature gravity can also be obtained by using the approach of [87–89], respectively. The expression for the entropy of the third order Lovelock black hole given by (35) is the same as that of [89]). Thus, we may infer that the entropy of the black hole obtained from the first law of black hole mechanics is indeed the Wald entropy.

From (35) and (36), we find that the entropy is not proportional to the one-fourth of the horizon area as in the case of the black holes in the Einstein gravity. However, if we take the limit $\alpha' \to 0$, we can recover the usual area law of black hole entropy in the BI-AdS black hole [16] as

$$S = \frac{\pi^3}{4} r_+^5. \quad (37)$$

In our study of critical phenomena, we will be mainly concerned with the spherically symmetric space-time. In this regard we will always take the value of k to be $+1$. Substituting $k = 1$ in (32) and (35) we finally obtain the expressions for the "quasilocal energy," the Hawking temperature, and the entropy of the third order LBI-AdS black hole as

$$M = \frac{5\pi^2}{16}\left[\frac{\alpha'^2}{3} + r_+^4 + \alpha' r_+^2 + \frac{2b^2 r_+^6}{15} \times \left(1 - \sqrt{1+\eta_+} - \frac{\Lambda}{2b^2} + \frac{20Q^2}{b^2\pi^4 r_+^{10}}\mathscr{H}(\eta_+)\right)\right], \quad (38)$$

$$T = \frac{10r_+^4 + 5\alpha' r_+^2 + 2b^2 r_+^6\left(1 - \sqrt{1 + 16Q^2/b^2\pi^4 r_+^{10}}\right) - \Lambda r_+^6}{10\pi r_+ \left(r_+^2 + \alpha'\right)^2}, \quad (39)$$

$$S = \frac{\pi^3}{4}\left(r_+^5 + \frac{10\alpha' r_+^3}{3} + 5\alpha'^2 r_+\right). \quad (40)$$

3. Phase Transition and Stability of the Third-Order LBI-AdS Black Hole

In this section we aim to discuss the nature of phase transition and the stability of the third order LBI-AdS black hole. A powerful method, based on Ehrenfest's scheme of ordinary thermodynamics, was introduced by the authors of [28] in order to determine the nature of phase transition in black holes. Using this analytic method, phase transition phenomena in various AdS black holes were explored [29–32]. Also, phase transition in higher dimensional AdS black holes has been discussed in [15].

In this paper we have qualitatively discussed the phase transition phenomena in the third order LBI-AdS black hole following the arguments presented in the above mentioned works. At this point it must be stressed that the method presented in [28] has not yet been implemented for the present black hole. However, we will not present any quantitative discussion in this regard.

From the $T - r_+$ plot (Figures 1 and 2) it is evident that there is no discontinuity in the temperature of the black hole. This rules out the possibility of first order phase transition [15, 29–32].

In order to see whether there is any higher order phase transition, we calculate the specific heat of the black hole. In the canonical ensemble framework the specific heat at constant charge (this is analogous to the specific heat at constant volume (C_V) in the ordinary thermodynamics) (C_Q) can be calculated as [16, 53]

$$C_Q = T\left(\frac{\partial S}{\partial T}\right)_Q = T\frac{(\partial S/\partial r_+)_Q}{(\partial T/\partial r_+)_Q} = \frac{\mathscr{N}(r_+, Q)}{\mathscr{D}(r_+, Q)}, \quad (41)$$

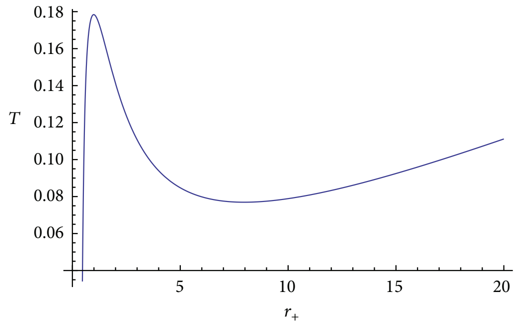

FIGURE 1: Plot of the Hawking temperature (T) against horizon radius (r_+), for $\alpha' = 0.5$, $Q = 0.50$, and $b = 10$.

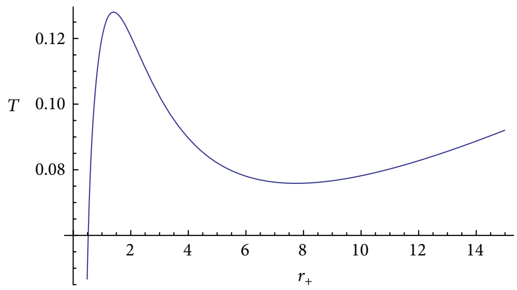

FIGURE 2: Plot of the Hawking temperature (T) against horizon radius (r_+), for $\alpha' = 1.0$, $Q = 0.50$, and $b = 10$.

where

$$\mathscr{N}(r_+, Q) = \frac{5}{4}\pi^7 r_+^5 \left(r_+^2 + \alpha'\right)^3 \sqrt{1 + \frac{16Q^2}{b^2\pi^4 r_+^{10}}} \times \left[10r_+^2 + 5\alpha' + 2b^2 r_+^4\left(1 - \sqrt{1 + \frac{16Q^2}{b^2\pi^4 r_+^{10}}}\right) - \Lambda r_+^4\right], \quad (42)$$

$$\mathscr{D}(r_+, Q) = 128Q^2 + \left[15\pi^4 r_+^6 \alpha' + 5\pi^4 r_+^4 \alpha'^2 - \Lambda\pi^4 r_+^{10} - 5\pi^4 r_+^8\left(2 + \alpha'\Lambda\right)\right]\sqrt{1 + \frac{16Q^2}{b^2\pi^4 r_+^{10}}} - \left(2b^2\pi^4 r_+^{10} + 10b^2\pi^4 r_+^8 \alpha'\right)\left(1 - \sqrt{1 + \frac{16Q^2}{b^2\pi^4 r_+^{10}}}\right). \quad (43)$$

In the derivation of (41) we have used (39) and (40).

In Figures 3, 4, 5, 6, 7, 8, 9, and 10, we have plotted C_Q against the horizon radius r_+ (here we have zoomed in the plots near the two critical points r_i ($i = 1, 2$) separately).

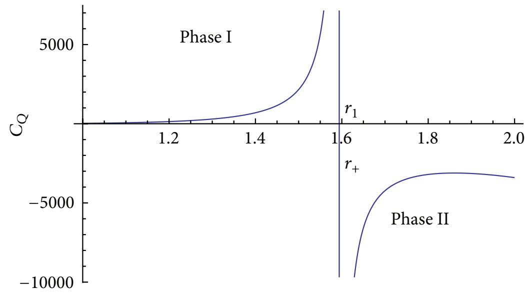

FIGURE 3: Plot of specific heat (C_Q) against horizon radius (r_+), for $\alpha' = 0.5$, $Q = 15$, and $b = 0.60$ at the critical point r_1.

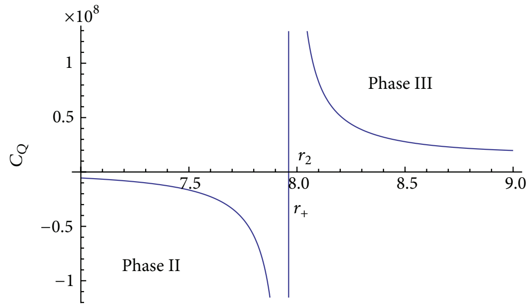

FIGURE 4: Plot of specific heat (C_Q) against horizon radius (r_+), for $\alpha' = 0.5$, $Q = 15$, and $b = 0.60$ at the critical point r_2.

The numerical values of the roots of (41) are given in Tables 1(a) and 1(b). For convenience we have written the real roots of (41) only. From our analysis it is observed that the specific heat always possesses simple poles. Moreover, there are two real positive roots (r_i $(i = 1, 2)$) of the denominator of C_Q for different values of the parameters b, Q, and α'. Also, from the $C_Q - r_+$ plots it is observed that the specific heat suffers discontinuity at the critical points r_i $(i = 1, 2)$. This property of C_Q allows us to conclude that at the critical points there is indeed a continuous higher order phase transition [16, 53].

Let us now see whether there is any bound in the values of the parameters b, Q, and α'. At this point it must be stressed that a bound in the parameter values (b, Q) for the Born-Infeld-AdS black holes in $(3 + 1)$-dimensions was found earlier [53, 57]. Moreover, this bound is removed if we consider space-time dimensions greater than four [16]. Thus, it will be very much interesting to check whether the third-order LBI-AdS black holes possess similar features. In order to do so, we will consider the extremal third order LBI-AdS black hole. In this case both $f(r)$ and df/dr vanish at the degenerate horizon r_e [16, 53, 57]. The above two conditions for extremality result in the following equation:

$$10r_e^4 + 5\alpha' r_e^2 + 2b^2 r_e^6\left(1 - \sqrt{1 + \frac{16Q^2}{b^2\pi^4 r_e^{10}}}\right) - \Lambda r_e^6 = 0. \tag{44}$$

In Tables 2(a) and 2(b) we give the numerical solutions of (44) for different choices of the values of the parameters b and Q for fixed values of α'. From this analysis we observe that for arbitrary choices of the parameters b and Q we always obtain atleast one real positive root of (44). This implies that there exists a smooth extremal limit for arbitrary b and Q and there is no bound on the parameter space for a particular value of α'. Thus, the result obtained here (regarding the bound in the parameter values) is in good agreement with that obtained in [16].

We will now analyse the thermodynamic stability of the third order LBI-AdS black hole. This is generally done by studying the behaviour of C_Q at the critical points [16, 30, 32, 53, 57]. The $C_Q - r_+$ plots show that there are indeed three phases of the black hole. These phases can be classified as, Phase I $(0 < r_+ < r_1)$, Phase II $(r_1 < r_+ < r_2)$, and Phase III, $(r_+ > r_2)$. Since the higher mass black hole possesses larger entropy/horizon radius, there is a phase transition at r_1 from smaller mass black hole (Phase I) to intermediate (higher mass) black hole (Phase II). The critical point r_2 corresponds to a phase transition from an intermediate (higher mass) black hole (Phase II) to a larger mass black hole (Phase III). Moreover, from the $C_Q - r_+$ plots we note that the specific heat C_Q is positive for Phase I and Phase III whereas it is negative for Phase II. Therefore Phase I and Phase III correspond to thermodynamically stable phases $(C_Q > 0)$, whereas Phase II corresponds to thermodynamically unstable phase $(C_Q < 0)$.

We can further extend our stability analysis by considering the free energy of the third order LBI-AdS black hole. The free energy plays an important role in the theory of phase transition and critical phenomena. We may define the free energy of the third order LBI-AdS black hole as

$$\mathcal{F}(r_+, Q) = M(r_+, Q) - TS. \tag{45}$$

Using (38), (39), and (40) we can write (45) as

$$\begin{aligned}\mathcal{F} &= \frac{5\pi^2}{16}\left[\frac{\alpha'^2}{3} + r_+^4 + \alpha' r_+^2 + \frac{2b^2 r_+^6}{15}\right. \\ &\quad\times\left(1 - \sqrt{1 + \frac{16Q^2}{b^2\pi^4 r_+^{10}}} - \frac{\Lambda}{2b^2}\right. \\ &\quad\left.\left. + \frac{20Q^2}{b^2\pi^4 r_+^{10}}\mathcal{H}\left(\frac{1}{2}, \frac{2}{5}, \frac{7}{5}, -\frac{16Q^2}{b^2\pi^4 r_+^{10}}\right)\right)\right] \\ &\quad - \frac{\pi^2\left(r_+^5 + \left(10\alpha' r_+^3/3\right) + 5\alpha'^2 r_+\right)}{40 r_+\left(r_+^2 + \alpha'\right)^2} \\ &\quad\times\left[10r_+^4 + 5\alpha' r_+^2 + 2b^2 r_+^6\left(1 - \sqrt{1 + \frac{16Q^2}{b^2\pi^4 r_+^{10}}}\right) - \Lambda r_+^6\right].\end{aligned} \tag{46}$$

In Figures 11, 12, 13, and 14 we have given the plots of the free energy ($\mathcal{F}$) of the black hole with the radius of the outer horizon r_+. The free energy ($\mathcal{F}$) has a minima $\mathcal{F} = \mathcal{F}_m$ at

Table 1: (a) Real roots (41) for $\alpha' = 0.5$ and $l = 10$. (b) Real roots of (41) for $\alpha' = 1.0$ and $l = 10$.

(a)

Q	b	r_1	r_2	r_3	r_4
15	0.6	1.59399	7.96087	−7.96087	−1.59399
8	0.2	1.18048	7.96088	−7.96088	−1.18048
5	0.5	1.25190	7.96088	−7.96088	−1.25190
0.8	20	1.01695	7.96088	−7.96088	−1.01695
0.3	15	0.975037	7.96088	−7.96088	−0.975037
0.5	10	0.989576	7.96088	−7.96088	−0.989576
0.5	1	0.989049	7.96088	−7.96088	−0.989049
0.5	0.5	0.987609	7.96088	−7.96088	−0.987609
0.05	0.05	0.965701	7.96088	−7.96088	−0.965701

(b)

Q	b	r_1	r_2	r_3	r_4
15	0.6	1.76824	7.74534	−7.74534	−1.76824
8	0.2	1.53178	7.74535	−7.74535	−1.53178
5	0.5	1.51668	7.74535	−7.74535	−1.51668
0.8	20	1.40553	7.74535	−7.74535	−1.40553
0.3	15	1.40071	7.74535	−7.74535	−1.40071
0.5	10	1.40214	7.74535	−7.74535	−1.40214
0.5	1	1.40214	7.74535	−7.74535	−1.40214
0.5	0.5	1.40213	7.74535	−7.74535	−1.40213
0.05	0.05	1.39992	7.74535	−7.74535	−1.39992

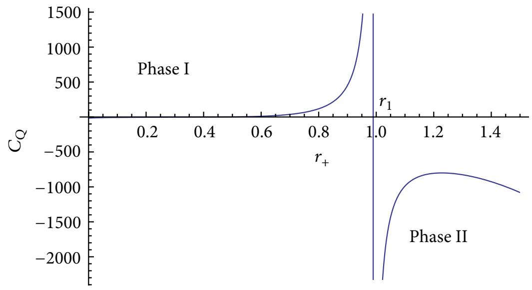

Figure 5: Plot of specific heat (C_Q) against horizon radius (r_+), for $\alpha' = 0.5$, $Q = 0.50$, and $b = 10$ at the critical point r_1.

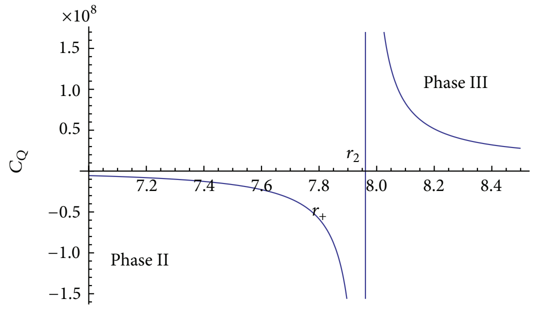

Figure 6: Plot of specific heat (C_Q) against horizon radius (r_+), for $\alpha' = 0.5$, $Q = 0.50$, and $b = 10$ at the critical point r_2.

$r_+ = r_m$. This point of minimum-free energy is exactly the same as the first critical point $r_+ = r_1$, where the black hole shifts from a stable to an unstable phase. On the other hand $\mathcal{F}$ has a maxima $\mathcal{F} = \mathcal{F}_0$ at $r_+ = r_0$. The point at which $\mathcal{F}$ reaches its maximum value is identical with the second critical point $r_+ = r_2$, where the black hole changes from unstable to stable phase. We can further divide the $\mathcal{F} - r_+$ plot into three distinct regions. In the first region $r_1' < r_+ < r_m$ the negative-free energy decreases until it reaches the minimum value ($\mathcal{F}_m$) at $r_+ = r_m$. This region corresponds to the stable phase (Phase I: $C_Q > 0$) of the black hole. The free energy changes its slope at $r_+ = r_m$ and continues to increase in the second region $r_m < r_+ < r_0$ approaching towards the maximum value ($\mathcal{F}_0$) at $r_+ = r_0$. This region corresponds to Phase II of the $C_Q - r_+$ plot, where the black hole becomes unstable ($C_Q < 0$). The free energy changes its slope once again at $r_+ = r_0$ and decreases to zero at $r_+ = r_2'$ and finally becomes negative for $r_+ > r_2'$. This region of the $\mathcal{F} - r_+$ plot corresponds to the Phase III of the $C_Q - r_+$ plot where the black hole finally becomes stable ($C_Q > 0$).

4. Critical Exponents and Scaling Hypothesis

In thermodynamics, the theory of phase transition plays a crucial role to understand the behavior of a thermodynamic system. The behavior of thermodynamic quantities near the critical point(s) of phase transition gives a considerable amount of information about the system. The behavior of a thermodynamic system near the critical point(s) is usually studied by means of a set of indices known as the critical exponents [36, 37]. These are generally denoted by a set of Greek letters: α, β, γ, δ, ϕ, ψ, η, and ν. The critical exponents describe the nature of singularities in various measurable thermodynamic quantities near the critical point(s).

In this section we aim to determine the first six static critical exponents (α, β, γ, δ, ϕ, and ψ). For this purpose we shall follow the method discussed in [16, 33, 53]. We shall then discuss the static scaling laws and static scaling hypothesis. We shall determine the other two critical exponents (ν and η) from two additional scaling laws.

Critical Exponent α. In order to determine the critical exponent α which is associated with the singularity of C_Q near the critical points r_i ($i = 1, 2$), we choose a point in the infinitesimal neighborhood of r_i as

$$r_+ = r_i (1 + \Delta), i = 1, 2, \tag{47}$$

where $|\Delta| \ll 1$. Let us denote the temperature at the critical point by $T(r_i)$ and define the quantity

$$\epsilon = \frac{T(r_+) - T(r_i)}{T(r_i)} \tag{48}$$

such that $|\epsilon| \ll 1$.

TABLE 2: (a) Roots of (44) for $\alpha' = 0.5$ and $l = 10$. (b) Roots of (44) for $\alpha' = 1.0$ and $l = 10$.

(a)

Q	b	r_{e1}^2	r_{e2}^2	$r_{e3,e4}^2$	r_{e5}^2
15	0.6	−66.4157	—	—	+0.632734
8	0.2	−66.4157	—	—	+0.121590
5	0.5	−66.4157	—	—	+0.200244
0.8	20	−66.4157	−0.408088	$-0.05881 \pm i0.304681$	+0.268514
0.3	15	−66.4157	−0.304225	$-0.0517567 \pm i0.175488$	+0.145685
0.5	10	−66.4157	−0.349764	$-0.0593449 \pm i0.240144$	+0.192519
0.5	1	−66.4157	—	—	+0.0221585
0.5	0.5	−66.4157	—	—	+0.00625335
0.05	0.05	−66.4157	—	—	$+6.57019 \times 10^{-7}$

(b)

Q	b	r_{e1}^2	r_{e2}^2	$r_{e3,e4}^2$	r_{e5}^2
15	0.6	−66.1628	—	—	+0.50473
8	0.2	−66.1628	—	—	+0.0546589
5	0.5	−66.1628	—	—	+0.110054
0.8	20	−66.1629	−0.563564	$-0.0913395 \pm i0.266887$	+0.236186
0.3	15	−66.1629	−0.514815	$-0.0584676 \pm i0.14609$	+0.116834
0.5	10	−66.1629	−0.531635	$-0.0760937 \pm i0.210702$	+0.155024
0.5	1	−66.1629	−0.542417	—	+0.00640486
0.5	0.5	−66.1629	—	—	+0.00163189
0.05	0.05	−66.1629	—	—	$+1.64256 \times 10^{-7}$

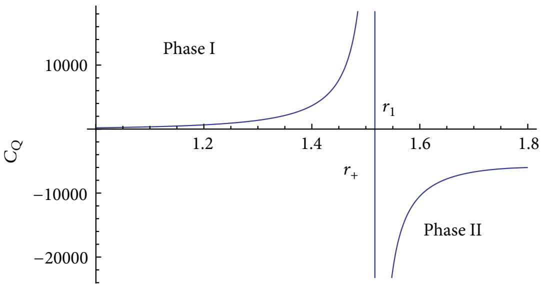

FIGURE 7: Plot of specific heat (C_Q) against horizon radius (r_+), for $\alpha' = 1.0$, $Q = 5$, and $b = 0.5$ at the critical point r_1.

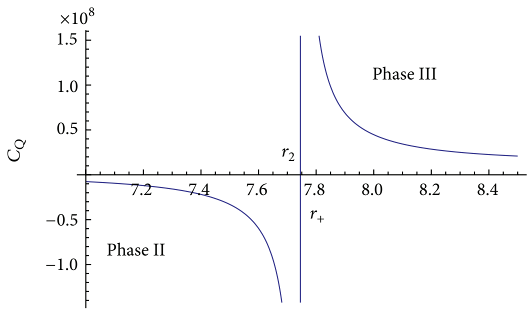

FIGURE 8: Plot of specific heat (C_Q) against horizon radius (r_+), for $\alpha' = 1.0$, $Q = 5$, and $b = 0.5$ at the critical point r_2.

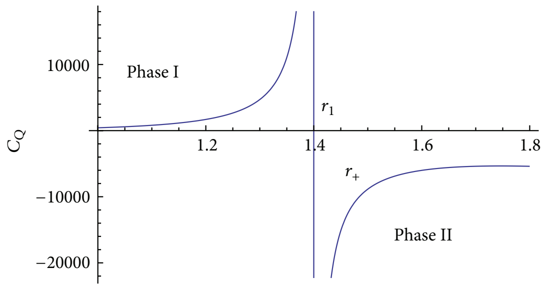

FIGURE 9: Plot of specific heat (C_Q) against horizon radius (r_+), for $\alpha' = 1.0$, $Q = 0.05$, and $b = 0.05$ at the critical point r_1.

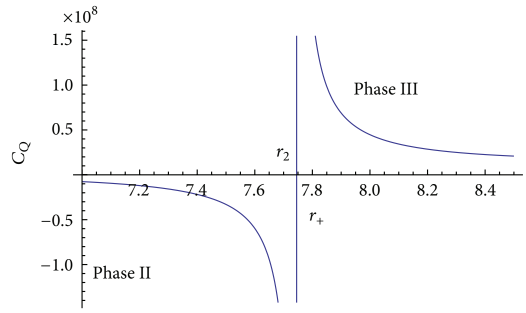

FIGURE 10: Plot of specific heat (C_Q) against horizon radius (r_+), for $\alpha' = 1.0$, $Q = 0.05$, and $b = 0.05$ at the critical point r_2.

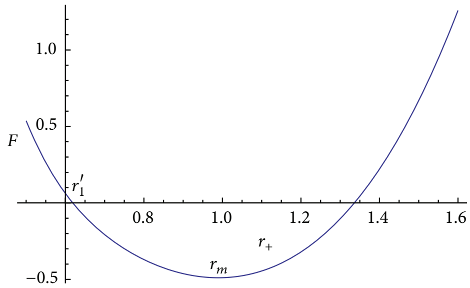

Figure 11: Plot of free energy ($\mathcal{F}$) against horizon radius (r_+), for $\alpha' = 0.50$, $Q = 0.50$, and $b = 10$.

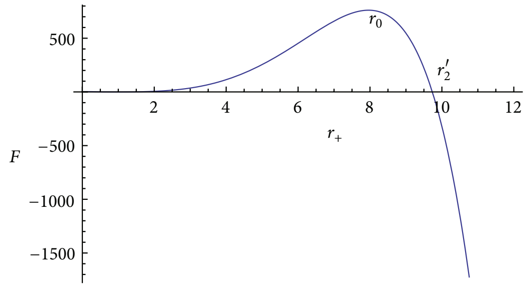

Figure 12: Plot of free energy ($\mathcal{F}$) against horizon radius (r_+), for $\alpha' = 0.50$, $Q = 0.50$, and $b = 10$.

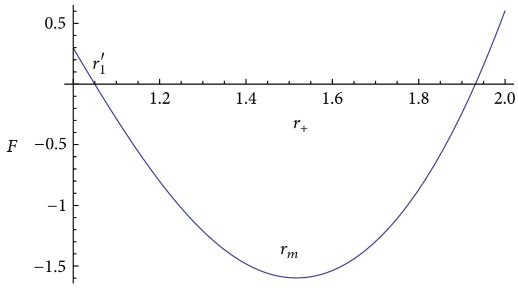

Figure 13: Plot of free energy ($\mathcal{F}$) against horizon radius (r_+), for $\alpha' = 1.0$, $Q = 5$, and $b = 0.5$.

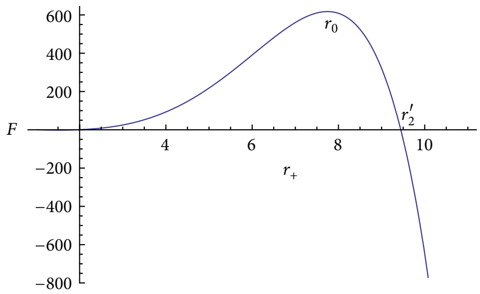

Figure 14: Plot of free energy ($\mathcal{F}$) against horizon radius (r_+), for $\alpha' = 1.0$, $Q = 5$, and $b = 0.5$.

We now Taylor expand $T(r_+)$ in the neighborhood of r_i keeping the charge constant ($Q = Q_c$), which yields

$$\begin{aligned} T(r_+) = {} & T(r_i) + \left[\left(\frac{\partial T}{\partial r_+}\right)_{Q=Q_c}\right]_{r_+=r_i} (r_+ - r_i) \\ & + \frac{1}{2}\left[\left(\frac{\partial^2 T}{\partial r_+^2}\right)_{Q=Q_c}\right]_{r_+=r_i} (r_+ - r_i)^2 + \cdots . \end{aligned} \tag{49}$$

Since the divergence of C_Q results from the vanishing of $(\partial T/\partial r_+)_Q$ at the critical point r_i (41), we may write (49) as

$$T(r_+) = T(r_i) + \frac{1}{2}\left[\left(\frac{\partial^2 T}{\partial r_+^2}\right)_{Q=Q_c}\right]_{r_+=r_i} (r_+ - r_i)^2, \tag{50}$$

where we have neglected the higher order terms in (49).

Using (47) and (48) we can finally write (50) as

$$\Delta = \frac{\epsilon^{1/2}}{\Gamma_i^{1/2}}, \tag{51}$$

where

$$\Gamma_i = \frac{r_i^2}{2T(r_i)}\left[\left(\frac{\partial^2 T}{\partial r_+^2}\right)_{Q=Q_c}\right]_{r_+=r_i}. \tag{52}$$

The detailed expression of Γ_i is very much cumbersome and we will not write it for the present work.

If we examine the $T-r_+$ plots (Figures 1 and 2), we observe that near the critical point $r_+ = r_1$ (which corresponds to the "hump") $T(r_+) < T(r_1)$ so that $\epsilon < 0$, and on the contrary, near the critical point $r_+ = r_2$ (which corresponds to the "dip") $T(r_+) > T(r_2)$ implying $\epsilon > 0$.

Substituting (47) into (41) we can write the singular part of C_Q as

$$C_Q = \frac{\mathcal{N}'(r_i, Q_c)}{\Delta \cdot \mathcal{D}'(r_i, Q_c)}, \tag{53}$$

where $\mathcal{N}'(r_i, Q_c)$ is the value of the numerator of C_Q (42) at the critical point $r_+ = r_i$ and critical charge $Q = Q_c$. The expression for $\mathcal{D}'(r_i, Q_c)$ is given by

$$\begin{aligned} \mathcal{D}'(r_i, Q_c) = {} & \mathcal{D}_1'(r_i, Q_c) + \mathcal{D}_2'(r_i, Q_c) \\ & + \mathcal{D}_3'(r_i, Q_c), \end{aligned} \tag{54}$$

where

$$\begin{aligned} \mathcal{D}_1'(r_i, Q_c) & \\ = {} & 10\pi^4 r_i^4 \sqrt{1 + \frac{16Q_c^2}{b^2\pi^4 r_i^{10}}} \\ & \times \left[\left(2\alpha'^2 + 2b^2 r_i^6 + 8b^2 r_i^4 \alpha'\right) \right. \\ & \left. - \left(\Lambda r_i^6 + 9\alpha' r_i^2 + 4\left(2 + \Lambda\alpha'\right) r_i^4\right)\right] \end{aligned}$$

$$\mathscr{D}_2'(r_i, Q_c) = -\frac{80Q_c^2}{b^2 r_i^2}\left[\frac{15\alpha'}{r_i^2} + \frac{5\alpha'^2}{r_i^4} - \Lambda r_i^2 - 5\left(2 + \Lambda\alpha'\right)\right]$$

$$\begin{aligned}\mathscr{D}_3'(r_i, Q_c) &\\ &= -20b^2\pi^4 r_i^8\left[r_i^2\left(1 + \frac{8Q_c^2}{b^2\pi^4 r_i^{10}}\right)\right.\\ &\left.\quad + 4\alpha'\left(1 + \frac{10Q_c^2}{b^2\pi^4 r_i^{10}}\right)\right].\end{aligned} \tag{55}$$

It is to be noted that while expanding the denominator of C_Q, we have retained the terms which are linear in Δ, and all other higher order terms of Δ have been neglected.

Using (53) we may summarize the critical behavior of C_Q near the critical points (r_1 and r_2) as follows:

$$C_Q \sim \begin{cases}\left[\dfrac{\mathscr{A}_i}{(-\epsilon)^{1/2}}\right]_{r_i=r_1} & \epsilon < 0\\ \left[\dfrac{\mathscr{A}_i}{(+\epsilon)^{1/2}}\right]_{r_i=r_2} & \epsilon > 0,\end{cases} \tag{56}$$

where

$$\mathscr{A}_i = \frac{\Gamma_i^{1/2}\mathscr{N}'(r_i, Q_c)}{\mathscr{D}'(r_i, Q_c)}. \tag{57}$$

We can combine the right-hand side of (56) into a single expression, which describes the singular nature of C_Q near the critical point r_i, yielding

$$\begin{aligned}C_Q &= \frac{\mathscr{A}_i}{|\epsilon|^{1/2}}\\ &= \frac{\mathscr{A}_i T_i^{1/2}}{|T - T_i|^{1/2}},\end{aligned} \tag{58}$$

where we have used (48). Here T and T_i are the abbreviations of $T(r_+)$ and $T(r_i)$, respectively.

We can now compare (58) with the standard form

$$C_Q \sim |T - T_i|^{-\alpha} \tag{59}$$

which gives $\alpha = 1/2$.

Critical Exponent β. The critical exponent β is related to the electric potential at infinity (Φ) by the relation

$$\Phi(r_+) - \Phi(r_i) \sim |T - T_i|^{\beta}, \tag{60}$$

where the charge (Q) is kept constant.

Near the critical point $r_+ = r_i$ the Taylor expansion of $\Phi(r_+)$ yields

$$\Phi(r_+) = \Phi(r_i) + \left[\left(\frac{\partial\Phi}{\partial r_+}\right)_{Q=Q_c}\right]_{r_+=r_i}(r_+ - r_i) + \cdots \tag{61}$$

Neglecting the higher order terms and using (30) and (51) we may rewrite (61) as

$$\begin{aligned}\Phi(r_+) - \Phi(r_i) &\\ &= -\left(\frac{4Q_c}{\pi^2 r_i^4\Gamma_i^{1/2}T_i^{1/2}\sqrt{1 + 16Q_c^2/b^2\pi^4 r_i^{10}}}\right)|T - T_i|^{1/2}.\end{aligned} \tag{62}$$

Comparing (62) with (60) we finally obtain $\beta = 1/2$.

Critical Exponent γ. We will now determine the critical exponent γ which is associated with the singularity of the inverse of the isothermal compressibility (κ_T^{-1}) at constant charge $Q = Q_c$ near the critical point $r_+ = r_i$ as

$$\kappa_T^{-1} \sim |T - T_i|^{-\gamma}. \tag{63}$$

In order to calculate κ_T^{-1} we use the standard thermodynamic definition

$$\begin{aligned}\kappa_T^{-1} &= Q\left(\frac{\partial\Phi}{\partial Q}\right)_T\\ &= -Q\left(\frac{\partial\Phi}{\partial T}\right)_Q\left(\frac{\partial T}{\partial Q}\right)_\Phi,\end{aligned} \tag{64}$$

where in the last line of (64) we have used the identity

$$\left(\frac{\partial\Phi}{\partial T}\right)_Q\left(\frac{\partial T}{\partial Q}\right)_\Phi\left(\frac{\partial Q}{\partial\Phi}\right)_T = -1. \tag{65}$$

Using (30) and (39) we can write (64) as

$$\kappa_T^{-1} = \frac{\Omega(r_+, Q)}{\mathscr{D}(r_+, Q)}, \tag{66}$$

where $\mathscr{D}(r_+, Q)$ is the denominator identically equal to (43) (the denominator of C_Q), and the expression for $\Omega(r_+, Q)$ may be written as

$$\begin{aligned}\Omega(r_+, Q) &\\ &= \frac{Q}{5\pi^2 r_+^4}\\ &\quad\times\left[128Q^2 + \left(15\pi^4 r_+^6\alpha + 5\pi^4 r_+^4\alpha^2\right.\right.\\ &\quad\left. - \Lambda\pi^4 r_+^{10} - 5\pi^4 r_+^8(2 + \alpha\Lambda)\right)\\ &\quad\times\sqrt{1 + \frac{16Q^2}{b^2\pi^4 r_+^{10}}} - \left(2b^2\pi^4 r_+^{10} + 10b^2\pi^4 r_+^8\alpha\right)\\ &\quad\times\left(1 - \sqrt{1 + \frac{16Q^2}{b^2\pi^4 r_+^{10}}}\right)\\ &\quad\left.\times\mathscr{H}\left(\frac{1}{2}, \frac{2}{5}, \frac{7}{5}, -\frac{16Q^2}{b^2\pi^4 r_+^{10}}\right)\right] - \Sigma(r_+, Q),\end{aligned} \tag{67}$$

where

$$\Sigma(r_+,Q) = \frac{4Q}{5}\left[2b^2\pi^2 r_+^4\left(r_+^2+5\alpha\right)\sqrt{1+\frac{16Q^2}{b^2\pi^4 r_+^{10}}} + \pi^2\left(-15r_+^2\alpha - 5\alpha^2 + r_+^6\left(\Lambda - 2b^2\right) + 5r_+^4\left(2+\Lambda\alpha - 2b^2\alpha\right)\right)\right]. \tag{68}$$

From (66) we observe that κ_T^{-1} possesses simple poles. Moreover κ_T^{-1} and C_Q exhibit common singularities.

We are now interested in the behavior of κ_T^{-1} near the critical point $r_+ = r_i$. In order to do so we substitute (47) into (66). The resulting equation for the singular part of κ_T^{-1} may be written as

$$\kappa_T^{-1} = \frac{\Omega'(r_i,Q_c)}{\Delta\cdot\mathscr{D}'(r_i,Q_c)}. \tag{69}$$

In (69), $\Omega'(r_i,Q_c)$ is the value of the numerator of κ_T^{-1} (67) at the critical point $r_+ = r_i$ and critical charge $Q = Q_c$, whereas $\mathscr{D}'(r_i,Q_c)$ was identified earlier (54).

Substituting (51) in (69) we may express the singular nature of κ_T^{-1} near the critical points (r_1 and r_2) as

$$\kappa_T^{-1} \simeq \begin{cases}\left[\dfrac{\mathscr{B}_i}{(-\epsilon)^{1/2}}\right]_{r_i=r_1} & \epsilon<0\\ \left[\dfrac{\mathscr{B}_i}{(+\epsilon)^{1/2}}\right]_{r_i=r_2} & \epsilon>0,\end{cases} \tag{70}$$

where

$$\mathscr{B}_i = \frac{\Gamma_i^{1/2}\Omega'(r_i,Q_c)}{\mathscr{D}'(r_i,Q_c)}. \tag{71}$$

Combining the right-hand side of (70) into a single expression as before, we can express the singular behavior of κ_T^{-1} near the critical point r_i as

$$\kappa_T^{-1} = \frac{\mathscr{B}_i}{|\epsilon|^{1/2}} = \frac{\mathscr{B}_i T_i^{1/2}}{|T-T_i|^{1/2}}. \tag{72}$$

Comparing (72) with (63) we find $\gamma = 1/2$.

Critical Exponent δ. Let us now calculate the critical exponent δ which is associated with the electrostatic potential (Φ) for the fixed value $T = T_i$ of temperature. The relation can be written as

$$\Phi(r_+) - \Phi(r_i) \sim |Q-Q_i|^{1/\delta}. \tag{73}$$

In this relation Q_i is the value of charge (Q) at the critical point r_i. In order to obtain δ we first Taylor expand $Q(r_+)$ around the critical point $r_+ = r_i$. This yields

$$Q(r_+) = Q(r_i) + \left[\left(\frac{\partial Q}{\partial r_+}\right)_{T=T_i}\right]_{r_+=r_i}(r_+-r_i) + \frac{1}{2}\left[\left(\frac{\partial^2 Q}{\partial r_+^2}\right)_{T=T_i}\right]_{r_+=r_i}(r_+-r_i)^2 + \cdots. \tag{74}$$

Neglecting the higher order terms we can write (74) as

$$Q(r_+) - Q(r_i) = \frac{1}{2}\left[\left(\frac{\partial^2 Q}{\partial r_+^2}\right)_T\right]_{r_+=r_i}(r_+-r_i)^2. \tag{75}$$

Here we have used the standard thermodynamic identity

$$\left[\left(\frac{\partial Q}{\partial r_+}\right)_T\right]_{r_+=r_i}\left[\left(\frac{\partial r_+}{\partial T}\right)_Q\right]_{r_+=r_i}\left(\frac{\partial T}{\partial Q}\right)_{r_+=r_i} = -1 \tag{76}$$

and considered the fact that at the critical point $r_+ = r_i$, $(\partial T/\partial r_+)_Q$ vanishes.

Let us now define a quantity

$$\Upsilon = \frac{Q(r_+)-Q_i}{Q_i} = \frac{Q-Q_i}{Q_i}, \tag{77}$$

where $|\Upsilon| \ll 1$. Here we denote $Q(r_+)$ and $Q(r_i)$ by Q and Q_i, respectively.

Using (47) and (77) we obtain from (75)

$$\Delta = \frac{\Upsilon^{1/2}}{\Psi_i^{1/2}}\left[\frac{2Q_i}{r_i^2}\right]^{1/2}, \tag{78}$$

where

$$\Psi_i = \left[\left(\frac{\partial^2 Q}{\partial r_+^2}\right)_T\right]_{r_+=r_i}. \tag{79}$$

The expression for Ψ_i is very much cumbersome, and we shall not write it here.

We shall now consider the functional relation

$$\Phi = \Phi(r_+,Q) \tag{80}$$

from which we may write

$$\left[\left(\frac{\partial \Phi}{\partial r_+}\right)_T\right]_{r_+=r_i} = \left[\left(\frac{\partial \Phi}{\partial r_+}\right)_Q\right]_{r_+=r_i} + \left[\left(\frac{\partial Q}{\partial r_+}\right)_T\right]_{r_+=r_i}\left(\frac{\partial \Phi}{\partial Q}\right)_{r_+=r_i}. \tag{81}$$

Using (76) we can rewrite (81) as

$$\left[\left(\frac{\partial \Phi}{\partial r_+}\right)_{T=T_i}\right]_{r_+=r_i} = \left[\left(\frac{\partial \Phi}{\partial r_+}\right)_{Q=Q_c}\right]_{r_+=r_i}. \tag{82}$$

Now the Taylor expansion of Φ at constant temperature around $r_+ = r_i$ yields

$$\Phi(r_+) = \Phi(r_i) + \left[\left(\frac{\partial \Phi}{\partial r_+}\right)_{T=T_i}\right]_{r_+=r_i} (r_+ - r_i), \quad (83)$$

where we have neglected all the higher order terms.

Finally using (78), (82), and (30) we may write (83) as

$$\Phi(r_+) - \Phi(r_i) = \left(\frac{-4Q_c}{\pi^2 r_i^5 \sqrt{1 + 16Q_c^2/b^2\pi^4 r_i^{10}}}\right)\left(\frac{2}{\Psi_i}\right)^{1/2} |Q - Q_i|^{1/2}. \quad (84)$$

Comparing (73) and (84) we find that $\delta = 2$.

Critical Exponent ϕ. The critical exponent ϕ is associated with the divergence of the specific heat at constant charge (C_Q) at the critical point $r_+ = r_i$ as,

$$C_Q \sim |Q - Q_i|^{-\phi}. \quad (85)$$

Now from (53) and (58) we note that

$$C_Q \sim \frac{1}{\Delta} \quad (86)$$

which may be written as

$$C_Q \sim \frac{1}{|Q - Q_i|^{1/2}}, \quad (87)$$

where we have used (78).

Comparison of (87) with (85) yields $\phi = 1/2$.

Critical Exponent ψ. In order to calculate the critical exponent ψ, which is related to the entropy of the third order LBI-AdS black hole, we Taylor expand the entropy ($S(r_+)$) around the critical point $r_+ = r_i$. This gives

$$S(r_+) = S(r_i) + \left[\left(\frac{\partial S}{\partial r_+}\right)\right]_{r_+=r_i} (r_+ - r_i) + \cdots. \quad (88)$$

If we neglect all the higher order terms and use (40), (47), and (78), we can write (88) as

$$S(r_+) - S(r_i) = \frac{5\pi^3}{4}\left(r_i^4 + 2\alpha' r_i^2 + \alpha'^2\right)\left(\frac{2}{\Psi_i}\right)^{1/2} |Q - Q_i|^{1/2}. \quad (89)$$

Comparing (89) with the standard relation

$$S(r_+) - S(r_i) \sim |Q - Q_c|^{\psi}, \quad (90)$$

we finally obtain $\psi = 1/2$.

In Table 3 we write all the six critical exponents obtained from our analysis in a tabular form. For comparison we also give the critical exponents associated with some well-known systems.

Thermodynamic Scaling Laws and Static Scaling Hypothesis. The discussion of critical phenomena is far from complete unless we make a comment on the thermodynamic scaling laws. In standard thermodynamic systems the critical exponents are found to satisfy some relations among themselves. These relations are called thermodynamic scaling laws [36, 37]. These scaling relations are given as

$$\begin{gathered} \alpha + 2\beta + \gamma = 2 \\ \alpha + \beta(\delta + 1) = 2 \\ \phi + 2\psi - \frac{1}{\delta} = 1 \\ \beta(\delta - 1) = \gamma \\ (2 - \alpha)(\delta - 1) = \gamma(1 + \delta) \\ 1 + (2 - \alpha)(\delta\psi - 1) = (1 - \alpha)\delta. \end{gathered} \quad (91)$$

From the values of the critical exponents obtained in our analysis it is interesting to observe that all these scaling relations are indeed satisfied for the third order LBI-AdS black holes.

We shall now explore the static scaling hypothesis [36–38] for the third order LBI-AdS black hole. Since we are working in the canonical ensemble framework, the thermodynamic potential of interest is the *Helmholtz free energy*, $\mathcal{F}(T, Q) = M - TS$, where the symbols have their usual meaning.

Now the static scaling hypothesis states that, *close to the critical point the singular part of the Helmholtz free energy is a generalized homogeneous function of its variables*.

This asserts that there exist two parameters a_ϵ and a_Υ such that

$$\mathcal{F}(\lambda^{a_\epsilon}\epsilon, \lambda^{a_\Upsilon}\Upsilon) = \lambda\mathcal{F}(\epsilon, \Upsilon) \quad (92)$$

for any arbitrary number λ.

In an attempt to find the values of the scaling parameters a_ϵ and a_Υ, we shall now Taylor expand the Helmhotz free energy $\mathcal{F}(T, Q)$ near the critical point $r_+ = r_i$. This may be written as

$$\begin{aligned} \mathcal{F}(T, Q) &= \mathcal{F}(T, Q)\big|_{r_+=r_i} + \left[\left(\frac{\partial \mathcal{F}}{\partial T}\right)_Q\right]_{r_+=r_i} \\ &\times (T - T_i) + \frac{1}{2}\left[\left(\frac{\partial^2 \mathcal{F}}{\partial T^2}\right)_Q\right]_{r_+=r_i} \times (T - T_i)^2 \\ &+ \left[\left(\frac{\partial \mathcal{F}}{\partial Q}\right)_T\right]_{r_+=r_i} (Q - Q_i) + \frac{1}{2}\left[\left(\frac{\partial^2 \mathcal{F}}{\partial Q^2}\right)_T\right]_{r_+=r_i} \\ &\times (Q - Q_i)^2 + \left[\left(\frac{\partial^2 \mathcal{F}}{\partial T \partial Q}\right)\right]_{r_+=r_i} (T - T_i)(Q - Q_i) + \cdots. \end{aligned} \quad (93)$$

Table 3

Critical exponents	3rd-order LBI-AdS black hole	$CrBr_3$*	2D Ising model*	van der Waal system‡
α	0.5	0.05	0	0
β	0.5	0.368	0.125	0.5
γ	0.5	1.215	1.7	1.0
δ	2.0	4.28	15	3.0
ψ	0.5	0.60	—	—
ϕ	0.5	0.03	—	—

*These are the nonmean field values.
‡These values are taken from [46].

From (93) we can identify the second derivatives of $\mathcal{F}$ as

$$\left(\frac{\partial^2 \mathcal{F}}{\partial T^2}\right)_Q = \frac{-C_Q}{T},$$
$$\left(\frac{\partial^2 \mathcal{F}}{\partial Q^2}\right)_T = \frac{\kappa_T^{-1}}{Q}. \quad (94)$$

Note that since both C_Q and κ_T^{-1} diverge at the critical point, these derivatives can be justified as the singular parts of the free energy $\mathcal{F}$.

Since in the theory of critical phenomena we are mainly interested in the singular part of the relevant thermodynamic quantities, we sort out the singular part of $\mathcal{F}(T, Q)$ from (93), which may be written as

$$\begin{aligned}\mathcal{F}_s &= \frac{1}{2}\left[\left(\frac{\partial^2 \mathcal{F}}{\partial T^2}\right)_Q\right]_{r_+=r_i}(T-T_i)^2 \\ &\quad + \frac{1}{2}\left[\left(\frac{\partial^2 \mathcal{F}}{\partial Q^2}\right)_T\right]_{r_+=r_i}(Q-Q_i)^2 \\ &= \frac{-C_Q}{2T_i}(T-T_i)^2 + \frac{\kappa_T^{-1}}{2Q_i}(Q-Q_i)^2,\end{aligned} \quad (95)$$

where the subscript "s" denotes the singular part of the free energy $\mathcal{F}$.

Using (51), (58), (72), and (78) we may write the singular part of the Helmhotz free energy ($\mathcal{F}$) as

$$\mathcal{F}_s = \sigma_i \epsilon^{3/2} + \tau_i \Upsilon^{3/2}, \quad (96)$$

where

$$\sigma_i = \frac{-\mathcal{A}_i T_i}{2}, \qquad \tau_i = \frac{\mathcal{B}_i \Psi_i^{1/2} Q_i^{1/2} r_i}{2^{3/2}\Gamma_i^{1/2}}. \quad (97)$$

From (92) and (96) we observe that

$$a_\epsilon = a_\Upsilon = \frac{2}{3}. \quad (98)$$

This is an interesting result in the sense that, in general, a_ϵ and a_Υ are different for a generalized homogeneous function (GHF), but in this particular model of the black hole, these two scaling parameters are indeed identical. With this result we can argue that the Helmhotz free energy is usual homogeneous function for the third order LBI-AdS black hole. Moreover, we can determine the critical exponents $(\alpha, \beta, \gamma, \delta, \phi, \psi)$ once we calculate the scaling parameters. This is because these critical exponents are related to the scaling parameters as [36, 37]

$$\begin{aligned}\alpha &= 2 - \frac{1}{a_\epsilon}, & \beta &= \frac{1-a_\Upsilon}{a_\epsilon}, \\ \gamma &= \frac{2a_\Upsilon - 1}{a_\epsilon}, & \delta &= \frac{a_\Upsilon}{1 a_\Upsilon}, \\ \phi &= \frac{2a_\epsilon - 1}{a_\Upsilon}, & \psi &= \frac{1-a_\epsilon}{a_\Upsilon}.\end{aligned} \quad (99)$$

There are two other critical exponents associated with the behavior of the *correlation function* and *correlation length* of the system near the critical surface. We shall denote these two critical exponents as η and ν, respectively. If $G(\vec{r}_+)$ and ξ are the correlation function and the correlation length, respectively, we can relate η and ν with them as

$$G(\vec{r}_+) \sim r_+^{2-n-\eta},$$
$$\xi \sim |T - T_i|^{-\nu}. \quad (100)$$

For the time being we shall assume that the two additional scaling relations [36]

$$\gamma = \nu(2-\eta), \qquad (2-\alpha) = \nu n \quad (101)$$

hold for the third order LBI-AdS black hole. Using these two relations (101) and the values of α and γ, the exponents ν and η are found to be 1/4 and 0, respectively.

Although we have calculated η and ν assuming the additional scaling relations to be valid, it is not proven yet that these scaling relations are indeed valid for the black holes. One may adapt different techniques to calculate η and ν, but till now no considerable amount of progress has been made in this direction. One may compute these two exponents directly from the correlation of scalar modes in the theory of gravitation [48], but the present theories of critical phenomena in black holes are far from complete.

5. Conclusions

In this paper we have analyzed the critical phenomena in higher curvature charged black holes in a canonical framework. For this purpose we have considered the third order Lovelock-Born-Infeld-AdS (LBI-AdS) black holes in a spherically symmetric space-time. We systematically derived the thermodynamic quantities for such black holes. We are able to show that some of the thermodynamic quantities (C_Q, κ_T^{-1}) diverge at the critical points. From the nature of the plots we argue that there is a higher order phase transition in this black hole. Although the analytical estimation of the critical points is not possible due to the complexity of the relevant equations, we are able to determine the critical points numerically. However, all the critical exponents are calculated analytically near the critical points. Unlike the AdS black holes in the Einstein gravity, one interesting property of the higher curvature black holes is that the usual area law of entropy does not hold for these black holes. One might then expect that the critical exponents may differ from those for the AdS black holes in the Einstein gravity. But we find that all the critical exponents in the third order LBI-AdS black hole are indeed identical with those obtained in Einstein gravity [16, 53]. From this observation we may conclude that these black holes belong to the same universality class. Moreover, the critical exponents take the mean field values. It is to be noted that these black holes have distinct set of critical exponents which does not match the critical exponents of any other known thermodynamic systems. Another point that must be stressed is that the static critical exponents are independent of the spatial dimensionality of the AdS space-time. This suggests the mean field behavior in black holes as thermodynamic systems and allows us to study the phase transition phenomena in the black holes. We have also discussed the static scaling laws and static scaling hypothesis. The static critical exponents are found to satisfy the static scaling laws near the critical points. We have checked the consistency of the static scaling hypothesis. Apart from this we note that the two scaling parameters have identical values. This allows us to conclude that the Helmhotz free energy is indeed a homogeneous function for this type of black hole. We have determined the two other critical exponents ν and η associated with the correlation length (ξ) and correlation function ($G(\vec{r}_+)$) near the critical surface assuming the validity of the additional scaling laws. The values of these two exponents are found to be 1/4 and 0, respectively, in the six spatial dimensions. Although the other six critical exponents are independent of the spatial dimension of the system, these two exponents are very much dimension dependent.

In our analysis we have been able to resolve a number of vexing issues concerning the critical phenomena in third order LBI-AdS black holes. But there still remains some unsolved problems that encourage one to make further investigations into the system. First of all, we have made a qualitative argument about the nature of the phase transition in this black hole. One needs to go through detailed algebraic analysis in order to determine the true order of the phase transition [28–32]. Secondly, we have calculated the values of the exponents ν and η assuming that the additional scaling relations hold for this black hole. But there is no evidence whether these two laws hold for the black hole [16, 53]. These scaling relations may or may not hold for the black hole. The dimension dependence of these two exponents (ν and η) makes the issue highly nontrivial in higher dimensions. A further attempt to determine η and ν may be based on Ruppeiner's prescription [90], where it is assumed that the absolute value of the thermodynamic scalar curvature ($|\mathscr{R}|$) is proportional to the correlation volume ξ^n:

$$|\mathscr{R}| \sim \xi^n, \tag{102}$$

where n is the spatial dimension of the black hole. Now if we can calculate $\mathscr{R}$ using the standard method [32, 91, 92], we can easily determine ξ from (102). Evaluating ξ around the critical point $r_+ = r_i$ as before, we can determine ν directly. It is then straightforward to calculate η by using (84). This alternative approach, based on Ruppeiner's prescription, to determine ν and η needs high mathematical rigor, and also the complexity in the determination of the scalar curvature ($\mathscr{R}$) in higher dimensions makes the issue even more challenging.

Apart from the above mentioned issues, it would also be highly nontrivial if we aim to investigate the AdS/CFT duality as an alternative approach to make further insight into the theory of critical phenomena in these black holes. The *renormalization group* method may be another alternative way to describe the critical phenomena in these black holes.

Finally, it would be nice if we can apply the behaviour of the third-order LBI-AdS black holes for understanding several issues related to the brane-world cosmology. Following the prescription of [93], we can embed a brane in the bulk third-order LBI-AdS black hole and study the corresponding Friedmann-Robertson-Walker (FRW) cosmology. As a possible extension of our analysis, the study of thermodynamic properties of such a brane will be interesting. Performing analysis in the same line as is done in this paper we may compute the critical exponents of the brane which can be helpful to understand thermodynamical phases of the brane. On top of that, we can then compare critical behaviour of the black hole with that of the brane. Further, we may check the correspondence between the entropy of this black hole and that of the dual conformal field theory (CFT) that lives on the brane (Cardy-Verlinde formula) through the CFT/FRW relation. We may further look for possible modifications of the Cardy-Verlinde formula which may shed light on several questionable issues regarding our universe. Also, the contribution(s) due to the nonlinear Born-Infeld field, appearing in our model, on the FRW equations can be studied. There is also a scope to study the effects of the third order Lovelock coefficient (α') on the induced brane matter from the bulk LBI-AdS black hole. In relation to this, the investigation of the validity of the dominant energy condition (DEC) and/or weak energy condition (WEC) may turn out to be an important issue.

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PHYSICAL REVIEW D **87,** 104001 (2013)

Holographic s-wave condensate with nonlinear electrodynamics: A nontrivial boundary value problem

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In this paper, considering the probe limit, we analytically study the onset of holographic s-wave condensate in the planar Schwarzschild-AdS background. Inspired by various low-energy features of string theory, in the present work we replace the conventional Maxwell action with a (nonlinear) Born-Infeld action which essentially corresponds to the higher-derivative corrections of the gauge fields. Based on a variational method which is commonly known as the Sturm-Liouville eigenvalue problem and considering a nontrivial asymptotic solution for the scalar field, we compute the critical temperature for the s-wave condensation. The results thus obtained analytically agree well with the numerical findings [J. Jing and S. Chen, Phys. Lett. B **686**, 68 (2010)]. As a next step, we extend our perturbative technique to compute the order parameter for the condensation. Interestingly, our analytic results are found to be of the same order as the numerical values obtained earlier.

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I. INTRODUCTION

For the past few years, the AdS/CFT duality [1,2], which provides an exact correspondence between a gravity theory in $(d+1)$-dimensional AdS space and a strongly coupled gauge theory living on d dimensions, has been extensively applied in order to describe various phenomena in usual condensed matter physics, including high-T_c superconductivity. The holographic description of s-wave superconductors basically consists of a charged planar AdS black hole minimally coupled to a complex scalar field. The formation of scalar hair below the critical temperature (T_c) triggers the superconductivity in the boundary field theory through the mechanism of spontaneous $U(1)$ symmetry breaking [3–7].

Besides the conventional framework of Maxwell electrodynamics, there is always a provision for incorporating nonlinear electrodynamics in various aspects of gravity theories. The theory of nonlinear electrodynamics was originally introduced in an attempt to remove certain discrepancies, such as the infinite self-energy of electrons for the Maxwell theory [8]. Recently, gravity theories with nonlinear electrodynamics have found profound applications due to its emergence in the low-energy limit of the heterotic string theory [9–11]. Gravity theories that include such nonlinear effects have been investigated extensively for the past several years [12–25]. As a result, several intriguing features regarding the properties of black holes, such as regular black hole solutions [12,13], validation of the zeroth and the first laws of black hole mechanics [16], different asymptotic behaviors of black hole solutions [20], etc., have emerged.

Among the various theories with nonlinear electrodynamics, it is the Born-Infeld (BI) theory that has attained renewed attention due to its several remarkable features. Perhaps the most interesting and elegant regime for the application of the BI electrodynamics is the string theory. The BI theory effectively describes the *low-energy* behavior of the D branes, which are basically (nonperturbative) solitonic objects in string theory [26,27]. The nonlinear theories have entered into the gauge theories via the "braneworld" scenario [28]. String theory requires the inclusion of gravity theories in order to describe some of its fundamental properties. In this regard it is indeed essential to connect nonlinear electrodynamics with gravity. One of the interesting properties of the BI theory is that the electric field is regular for a pointlike particle. The regular BI theory with finite energy gives the nonsingular solutions of the field equations. In fact, the BI electrodynamics is the only nonlinear electrodynamic theory with a sensible weak field limit [26,29]. Another intriguing feature of the BI theory is that it remains invariant under electromagnetic duality [28,30–34]. All the above mentioned features of the BI theory provide a motivation to study Einstein gravity as well as higher-curvature gravity theories coupled to BI electrodynamics [13,35–42]. BI electrodynamics coupled to anti–de Sitter (AdS) gravity exhibits close resemblance to the Reissner-Nordström-AdS blck holes [43,44]. Also, it is reassuring to note that in $(3+1)$ dimensions, the black hole solution with BI electrodynamics possesses a (lower) bound to the extremality of the BI-AdS black holes [23–25,45].

For the past couple of years, gravity theories with both linear (usual Maxwell case) as well as nonlinear electrodynamics have been extensively studied in the context of

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AdS/CFT superconductivity [46–70]. Surprisingly, it is observed that fewer efforts have been made to study nonlinear theories [62–70]. The analyses that have been performed so far are mostly based on numerical techniques. A systematic analytic approach is therefore lacking in this particular context.

Inspired by all the above mentioned facts, in the present paper we aim to study the onset of holographic s-wave condensate in the framework of BI electrodynamics. In fact, this issue has been investigated earlier in Ref. [62] using numerical techniques. In the present paper, we aim to investigate the onset of s-wave condensate based on an analytic technique which is popularly known as the Sturm-Liouville (SL) eigenvalue problem [47]. In the present analysis we adopt the boundary condition $\langle\mathcal{O}\rangle = \psi^+$ and $\langle\mathcal{O}\rangle = \psi^- = 0$, which implies that the conformal dimension of the condensation operator $\mathcal{O}$ in the boundary field theory is $\Delta_+ = 2$ [62]. This boundary condition seems to be quite nontrivial as far as analytic computation is concerned. This is simply due to the fact that the perturbative technique required to solve the differential equations does not work in a straightforward manner. However, we overcome this problem by adopting certain mathematical tricks and have successfully computed the onset of s-wave condensate. From our analysis, we have been (analytically) able to show that the critical temperature (T_c) indeed gets affected due to the presence of higher-derivative corrections to the usual Maxwell action. In fact, it is found to be decreasing with the increase in the value of the BI coupling parameter (b), which suggests the onset of harder condensation. It is also noteworthy that our analytic results are in good agreement with the existing numerical results [62]. It should be noted that all our calculations have been carried out in the probe limit [7,48].[1]

The organization of the paper is as follows: In Sec. II, the basic setup for holographic superconductors in the Schwarzschild-AdS background is given. In Sec. III, we perform the analytic calculations involved to determine the critical temperature (T_c) of the condensate. Section IV deals with the computation of the order parameter ($\langle\mathcal{O}\rangle$) for the condensation. The last section is devoted to the conclusion.

II. BASIC SETUP

To begin with, we consider a fixed planar Schwarzschild-AdS black hole background, which reads [46][2]

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(dx^2 + dy^2), \quad (1)$$

where the metric function is

$$f(r) = \left(r^2 - \frac{r_+^3}{r}\right), \quad (2)$$

r_+ being the horizon radius of the black hole. The Hawking temperature of the black hole may be written as

$$T = \frac{1}{4\pi}\left(\frac{\partial f(r)}{\partial r}\right)_{r=r_+} = \frac{3r_+}{4\pi}. \quad (3)$$

In the presence of an electric field and a complex scalar field ($\psi(r)$), we may write the corresponding Lagrangian density as

$$\mathcal{L} = \mathcal{L}_{BI} - |\nabla_\mu\psi - iqA_\mu\psi|^2 - m^2|\psi|^2, \quad (4)$$

where $\mathcal{L}_{BI}$ is the Born-Infeld Lagrangian density given by Ref. [62],

$$\mathcal{L}_{BI} = \frac{1}{b}\left(1 - \sqrt{1 + \frac{bF}{2}}\right). \quad (5)$$

Here $F \equiv F^{\mu\nu}F_{\mu\nu}$ and b is the Born-Infeld parameter. It is to be noted that in our approach, we shall investigate the effect of the higher-derivative corrections to the gauge field in the leading order; i.e., we shall keep only terms linear in b. Thus, the results of this paper are valid only in the leading order of b.

The equation of motion for the electromagnetic field tensor $F_{\mu\nu}$ can be written as

$$\partial_\mu\left(\frac{\sqrt{-g}F^{\mu\nu}}{\sqrt{1 + \frac{bF}{2}}}\right) = \mathcal{J}^\nu. \quad (6)$$

Considering the ansatz [46] $\psi = \psi(r)$ and $A_\mu = (\phi(r), 0, 0, 0)$, the equations of motion for the scalar field $\psi(r)$ and the electric scalar potential $\phi(r)$ may be written as[3]

$$\psi''(r) + \left(\frac{f'}{f} + \frac{2}{r}\right)\psi'(r) + \left(\frac{\phi^2(r)}{f^2} + \frac{2}{f}\right)\psi(r) = 0, \quad (7)$$

$$\phi''(r) + \frac{2}{r}(1 - b\phi'^2(r))\phi'(r) - \frac{2\psi^2(r)}{f}(1 - b\phi'^2(r))^{\frac{3}{2}} = 0. \quad (8)$$

The above set of equations [Eqs. (7) and (8)] are written in the radial coordinate r. In order to carry out an analytic computation, we define a new variable $z = \frac{r_+}{r}$. In this new variable, Eqs. (7) and (8) become

[1] In the probe limit, gravity and matter decouple and the backreaction of the matter fields (the charged gauge field and the charged massive scalar field) is suppressed in the neutral black hole background. This is done by rescaling the matter fields by the charge (q) of the scalar field and then taking the limit $q \to \infty$. This simplifies the problem without hindering the physical properties of the system.

[2] Without loss of generality, we can choose $l = 1$, which follows from the scaling properties of the equation of motion. Also, the gravitational constant is set to be unity ($G = 1$).

[3] In our analysis, we take $m^2 = -2$.

$$z\psi''(z) - \frac{2+z^3}{1-z^3}\psi'(z) + \left[z\frac{\phi^2(z)}{r_+^2(1-z^3)^2} + \frac{2}{z(1-z^3)}\right]\psi(z) = 0, \tag{9}$$

$$\phi''(z) + \frac{2bz^3}{r_+^2}\phi'^3(z) - \frac{2\psi^2(z)}{z^2(1-z^3)}\left(1 - \frac{bz^4}{r_+^2}\phi'^2(z)\right)^{\frac{3}{2}}\phi(z) = 0. \tag{10}$$

Since the above equations [Eqs. (9) and (10)] are second-order differential equations, in order to solve them we must know the corresponding boundary conditions. The regularity of ϕ and ψ at the horizon requires $\phi(z=1) = 0$ and $\psi(z=1) = \frac{3}{2}\psi'(z=1)$.

On the other hand, at spatial infinity, ϕ and ψ can be approximated as

$$\phi(z) \approx \mu - \frac{\rho}{r} = \mu - \frac{\rho}{r_+}z \tag{11}$$

and

$$\psi(z) \approx \frac{\psi^{(+)}}{r^{\Delta_+}} + \frac{\psi^{(-)}}{r^{\Delta_-}} = \frac{\psi^{(+)}}{r_+^{\Delta_+}}z^{\Delta_+} + \frac{\psi^{(-)}}{r_+^{\Delta_-}}z^{\Delta_-}, \tag{12}$$

where $\Delta_\pm = \frac{3}{2} \pm \sqrt{\frac{9}{4} + m^2}$ is the conformal dimension of the *condensation operator* $\mathcal{O}$ in the boundary field theory, and μ and ρ are interpreted as the chemical potential and charge density of the dual field theory. Since we have considered $m^2 < 0$ (which is above the Breitenlohner-Freedman bound [71,72]), we are left with the two different condensation operators of different dimensionality corresponding to the choice of quantization of the scalar field ψ in the bulk. In the present context, either $\psi^{(+)}$ or $\psi^{(-)}$ will act as a condensation operator while the other will act as a source. In the present work, we choose $\psi^{(+)} = \langle\mathcal{O}\rangle$ and $\psi^{(-)}$ as its source. Since we want the condensation to take place in the absence of any source, we set $\psi^{(-)} = 0$. At this point, it must be stressed that for the present choice of ψ, the analytic calculations of various entities near the critical point get notoriously difficult, and special care should be taken in order to carry out a perturbative analysis. In the present work, we focus to evade the above mentioned difficulties by adopting certain mathematical techniques. Our analysis indeed shows a good agreement with the numerical results existing in the literature [62].

III. s-WAVE CONDENSATE WITH NONTRIVIAL BOUNDARY CONDITION

With the above setup in place, we now move on to investigate the relation between the critical temperature of condensation and the charge density.

At the critical temperature T_c the scalar field ψ vanishes, so Eq. (10) becomes

$$\phi''(z) + \frac{2bz^3}{r_{+(c)}^2}\phi'^3(z) = 0. \tag{13}$$

The solution for this equation in the interval $[z, 1]$ reads [64]

$$\phi(z) = \lambda r_{+(c)}\xi(z), \tag{14}$$

where

$$\xi(z) = \int_z^1 \frac{d\tilde{z}}{\sqrt{1 + b\lambda^2\tilde{z}^4}}. \tag{15}$$

We shall perform a perturbative expansion of $b\lambda^2$ on the rhs of Eq. (15) and retain only the terms that are linear in b such that $b\lambda^2 = b\lambda_0^2 + O(b^2)$, where λ_0^2 is the value of λ^2 for $b = 0$. Now, for our particular choice of $\psi^{(i)}$ $(i = +, -)$, we have $\lambda_0^2 \approx 17.3$ [47]. Recalling that the existing values of b in the literature are $b = 0.1, 0.2, 0.3$ [62], we observe that $b\lambda_0^2 > 1$. Consequently, the binomial expansion of the denominator in Eq. (15) has to done carefully. The integration appearing in Eq. (15) is done for two ranges of values of z, one for $z \le \Lambda < 1$ and the other for $\Lambda \le z \le 1$, where Λ is such that $b\lambda_0^2 z^4|_{z=\Lambda} = 1$. At this stage, it is to be noted that $b\lambda_0^2 z^4 < 1$ for $z < \Lambda$, whereas on the other hand $b\lambda_0^2 z^4 > 1$ for $z > \Lambda$.

For the first case ($z \le \Lambda < 1$),

$$\begin{aligned}\xi(z) = \xi_1(z) &= \int_z^\Lambda \frac{d\tilde{z}}{\sqrt{1 + b\lambda_0^2\tilde{z}^4}} + \int_\Lambda^1 \frac{d\tilde{z}}{\sqrt{1 + b\lambda_0^2\tilde{z}^4}} \\ &\approx \int_z^\Lambda\left(1 - \frac{b\lambda_0^2\tilde{z}^4}{2}\right) + \frac{1}{\sqrt{b}\lambda_0}\int_\Lambda^1\left(\frac{1}{\tilde{z}^2} - \frac{1}{2b\lambda_0^2\tilde{z}^6}\right) \\ &= \left[\frac{9}{5}\Lambda - z + \frac{z^5}{10\Lambda^4} - \Lambda^2 + \frac{\Lambda^6}{10}\right].\end{aligned} \tag{16}$$

Similarly, in the range $\Lambda \le z \le 1$, we have

$$\begin{aligned}\xi(z) = \xi_2(z) &= \int_z^1 \frac{d\tilde{z}}{\sqrt{1 + b\lambda_0^2\tilde{z}^4}} \\ &\approx \frac{1}{\sqrt{b}\lambda_0}\int_z^1\left(\frac{1}{\tilde{z}^2} - \frac{1}{2b\lambda_0^2\tilde{z}^6}\right) \\ &= \frac{\Lambda^2}{z^5}\left[z^4(1-z) + \frac{\Lambda^4}{10}(z^5 - 1)\right].\end{aligned} \tag{17}$$

From Eq. (17), one may note that the boundary condition $\phi(1) = 0$ is indeed satisfied [$\xi_2(1) = 0$].

We may now express $\psi(z)$ near the boundary as

$$\psi(z) = \frac{\langle\mathcal{O}\rangle}{\sqrt{2}r_+^2}z^2\mathcal{F}(z) \tag{18}$$

with the condition $\mathcal{F}(0) = 1$ and $\mathcal{F}'(0) = 0$.

Using Eq. (18), we may write Eq. (9) as

$$\mathcal{F}''(z) - \frac{(5z^4 - 2z)}{z^2(1-z^3)}\mathcal{F}'(z) - \frac{4z^3}{z^2(1-z^3)}\mathcal{F}(z) + \lambda^2 \frac{\xi^2(z)}{(1-z^3)^2}\mathcal{F}(z) = 0. \quad (19)$$

This equation can be put in the Sturm-Liouville form as

$$[p(z)\mathcal{F}'(z)]' + q(z)\mathcal{F}(z) + \lambda^2 g(z)\mathcal{F}(z) = 0 \quad (20)$$

with the following identifications:

$$p(z) = z^2(1-z^3), \qquad q(z) = -4z^3, \qquad g(z) = \frac{z^2}{(1-z^3)}\xi^2(z) = \chi(z)\xi^2(z), \quad (21)$$

where $\chi(z) = \frac{z^2}{(1-z^3)}$.

Using Eq. (21), we may write the eigenvalue λ^2 as

$$\lambda^2 = \frac{\int_0^1 \{p(z)[\mathcal{F}'(z)]^2 - q(z)[\mathcal{F}(z)]^2\}dz}{\int_0^1 \{g(z)[\mathcal{F}(z)]^2\}dz} = \frac{\int_0^1 \{p(z)[\mathcal{F}'(z)]^2 - q(z)[\mathcal{F}(z)]^2\}dz}{\int_0^\Lambda \{\chi(z)\xi_1^2(z)[\mathcal{F}(z)]^2\}dz + \int_\Lambda^1 \{\chi(z)\xi_2^2(z)[\mathcal{F}(z)]^2\}dz}. \quad (22)$$

We now choose the trial function $\mathcal{F}(z)$ as [47][4]

$$\mathcal{F}(z) = 1 - \alpha z^2, \quad (23)$$

which satisfies the conditions $\mathcal{F}(0) = 1$ and $\mathcal{F}'(0) = 0$. This form of the trial function is also compatible with the boundary behavior of the scalar field $\psi(z)$ [Eq. (12)].

Let us now determine the eigenvalues for different values of the parameter b.

For $b = 0.1$, we obtain

$$\lambda^2 = 300.769 + \frac{2.27395\alpha - 5.19713}{0.0206043 + (0.00265985\alpha - 0.0119935)\alpha}, \quad (24)$$

which has a minima for $\alpha \approx 0.653219$. Therefore, from Eq. (22) we obtain

$$\lambda^2 \approx 33.8298. \quad (25)$$

The value of λ^2 obtained from the perturbative calculation justifies our approximation for computing the integral in Eq. (15) up to order b and neglecting terms of order b^2 and higher, since a term of order b^2 can be estimated to be smaller than a term of order b.

[4]The functions $\mathcal{F}(z)$ and $\xi(z)$ [Eq. (15)] do not appear in the numerical analysis, since in the numerical method Eqs. (9) and (13) are solved directly.

TABLE I. A comparison between analytic and numerical values for the coefficient (γ) of T_c corresponding to different values of b.

Values of b	$\gamma_{\text{numerical}}$	γ_{SL}
0.1	0.10072	0.099
0.2	0.08566	0.093
0.3	0.07292	0.089

Using Eq. (25), the critical temperature for condensation (T_c) in terms of the charge density (ρ) can be obtained as

$$T_c = \frac{3r_{+(c)}}{4\pi} = \gamma\sqrt{\rho} \approx 0.099\sqrt{\rho}, \quad (26)$$

where $\gamma = \frac{3}{4\pi\sqrt{\lambda}}$ is the coefficient of T_c. The value thus obtained analytically is indeed in very good agreement with the numerical result: $T_c = 0.10072\sqrt{\rho}$ [62]. Similarly, for the other values of the Born-Infeld parameter (b), we obtain the corresponding perturbative values for the coefficients of T_c which are presented in Table I below.

Before concluding this section, we would like to emphasize the subtlety of the analytic method adapted here. We employ a perturbative technique to compute the integral in Eq. (15) up to order b. This approximation is valid since we have investigated the effect of the higher-derivative corrections up to the leading order in the nonlinear parameter (b). However, due to the nature of the integrand of Eq. (15), we had to be careful in separating the integral into two regions in order to perform a binomial expansion of the integrand.

IV. ORDER PARAMETER FOR CONDENSATION

In this section, we aim to calculate the order parameter $\langle\mathcal{O}\rangle$ for the s-wave condensate in the boundary field theory. In order to do so, we need to consider the behavior of the gauge field ϕ near the critical temperature T_c. Substituting Eq. (18) into Eq. (10), we may find

$$\phi''(z) + \frac{2bz^3}{r_+^2}\phi'^3(z) = \frac{\mathcal{F}^2(z)z^2\langle\mathcal{O}\rangle^2}{r_+^4(1-z^3)}\left(1 - \frac{3bz^4\phi'^2(z)}{2r_+^2}\right)\phi(z) + \mathbf{O}(b^2). \quad (27)$$

It is to be noted that in the subsequent analysis only terms up to linear order in b have been considered.

As a next step, we expand $\phi(z)$ perturbatively in the small parameter $\langle\mathcal{O}\rangle^2/r_+^4$ as follows:

$$\frac{\phi(z)}{r_+} = \frac{\phi_0(z)}{r_+} + \frac{\langle\mathcal{O}\rangle^2}{r_+^4}\chi(z) + \text{higher order terms}, \quad (28)$$

where ϕ_0 is the solution of Eq. (13). Here $\chi(z)$ is some arbitrary function which satisfies the boundary condition

$$\chi(1) = \chi'(1) = 0. \quad (29)$$

Substituting Eq. (28) and Eq. (14), we may write Eq. (27) in terms of $\chi(z)$ as

$$\chi''(z) + 6b\lambda^2 z^3 \xi'^2(z)\chi'(z) = \lambda \frac{\mathcal{F}^2(z) z^2 \xi(z)}{(1-z^3)}\left(1 - \frac{3b\lambda^2 z^4 \xi'^2(z)}{2}\right). \tag{30}$$

Multiplying both sides of Eq. (30) by $e^{3b\lambda^2 z^4 \xi'^2(z)/2}$ and considering terms up to order b, we obtain[5]

$$\frac{d}{dz}(e^{3b\lambda^2 z^4 \xi'^2(z)/2}\chi'(z)) = e^{3b\lambda^2 z^4 \xi'^2(z)/2}\lambda \frac{\mathcal{F}^2(z) z^2 \xi(z)}{(1-z^3)}\left(1 - \frac{3b\lambda^2 z^4 \xi'^2(z)}{2}\right) = \lambda \frac{\mathcal{F}^2(z) z^2 \xi(z)}{(1-z^3)}. \tag{31}$$

Using the boundary condition [Eq. (29)] and integrating Eq. (31) in the interval $[0, 1]$, we finally obtain

$$\chi'(0) = -\lambda(\mathcal{A}_1 + \mathcal{A}_2), \tag{32}$$

where

$$\begin{aligned}
\mathcal{A}_1 &= \int_0^\Lambda \frac{\mathcal{F}^2(z) z^2 \xi_1(z)}{(1-z^3)} \quad \text{for } 0 \le z < \Lambda \\
&= \frac{1}{12600\Lambda^4}\Big\{-70\sqrt{3}\pi(-1 - 10\Lambda^4 + \alpha(-2 + \Lambda^4(10\alpha + 18(2+\alpha)\Lambda - 10(2+\alpha)\Lambda^2 + (2+\alpha)\Lambda^6))) \\
&\quad + \Lambda(126\Lambda(-5 + 98\Lambda^3) + 30\alpha(84 + \Lambda^3(21 + 2\Lambda^3(244 + 21\Lambda(-10 + \Lambda^4)))) \\
&\quad - 35\alpha^2\Lambda^2(12 + \Lambda^3(474 + \Lambda(-360 + \Lambda^2(94 + 9\Lambda(-10 + 4\Lambda + \Lambda^4)))))) - 420\log(1-\Lambda) \\
&\quad + 210\log(1 + \Lambda + \Lambda^2) + 210(2\sqrt{3}(-1 - 10\Lambda^4 + \alpha(-2 + \Lambda^4(10\alpha + 18(2+\alpha)\Lambda - 10(2+\alpha)\Lambda^2 \\
&\quad + (2+\alpha)\Lambda^6)))\tan^{-1}\left(\frac{1+2\Lambda}{\sqrt{3}}\right) - (-2\alpha - 10(1+\alpha^2)\Lambda^4 + 18(\alpha-2)\alpha\Lambda^5 - 10(\alpha-2)\alpha\Lambda^6 \\
&\quad + (\alpha-2)\alpha\Lambda^{10})(2\log(1-\Lambda) - \log(1+\Lambda+\Lambda^2)) - 2(\alpha^2 + 20\alpha\Lambda^4 + \Lambda^5(18 - 10\Lambda + \Lambda^5))\log(1-\Lambda^3))\Big\}
\end{aligned} \tag{33}$$

and

$$\begin{aligned}
\mathcal{A}_2 &= \int_\Lambda^1 \frac{\mathcal{F}^2(z) z^2 \xi_2(z)}{(1-z^3)} \quad \text{for } \Lambda < z \le 1 \\
&= \frac{\Lambda^2}{360}\Big\{4\sqrt{3}\pi(-10(1 + \alpha(4+\alpha)) + (-1 + 2\alpha(1+\alpha))\Lambda^4) + 12\sqrt{3}(10 + 10\alpha(4+\alpha) \\
&\quad + \Lambda^4 - 2\alpha(1+\alpha)\Lambda^4)\tan^{-1}\left(\frac{1+2\Lambda}{\sqrt{3}}\right) + 3(-12\alpha(-10 + \Lambda(20 - 10\Lambda + \Lambda^5 + \Lambda^3(-1 + \log 3))) \\
&\quad - 6(\Lambda^2 + \Lambda^4(-1 + \log 3)) + \alpha^2(110 + \Lambda(-120 + \Lambda^2(40 + 3\Lambda(-15 + 4\Lambda + \Lambda^4)))) - 60\log 3) \\
&\quad + 60\log 3 - 24\alpha\Lambda^4\log\Lambda + 6(-10 + \Lambda^4 + 2\alpha(5\alpha + \Lambda^4))\log(1 + \Lambda + \Lambda^2))\Big\},
\end{aligned} \tag{34}$$

where $\xi_1(z)$ and $\xi_2(z)$ were identified earlier [Eqs. (16) and (17)].

Now, from Eqs. (11) and (28), we may write

$$\frac{\mu}{r_+} - \frac{\rho}{r_+^2} z = \frac{\phi_0(z)}{r_+} + \frac{\langle\mathcal{O}\rangle^2}{r_+^4}\chi(z) = \lambda\xi(z) + \frac{\langle\mathcal{O}\rangle^2}{r_+^4}\left\{\chi(0) + z\chi'(0) + \frac{z^2}{2!}\chi''(0) + \cdots\right\}. \tag{35}$$

It is to be noted that while writing the rhs of Eq. (35), we have made a Taylor expansion of $\chi(z)$ around $z = 0$. Comparing the coefficients of z from both sides of Eq. (35), we obtain

[5]The detailed derivation of this equation is given in the Appendix.

TABLE II. Values of β [Eq. (39)] for different values of b.

Values of b	Values of α	$(\mathcal{A}_1 + \mathcal{A}_2)$	β_{SL}	$\beta_{numerical}$
0.1	0.653219	0.0442811	117.919	207.360
0.2	0.656050	0.0388491	125.893	302.760
0.3	0.660111	0.0352282	132.205	432.640

$$\frac{\rho}{r_+^2} = \lambda - \frac{\langle \mathcal{O} \rangle^2}{r_+^4} \chi'(0). \tag{36}$$

By substituting Eq. (32), we may write Eq. (36) in the following form:

$$\frac{\rho}{r_+^2} = \lambda \left\{ 1 + \frac{\langle \mathcal{O} \rangle^2}{r_+^4} (\mathcal{A}_1 + \mathcal{A}_2) \right\}. \tag{37}$$

Substituting $\lambda = \rho / r_{+(c)}^2$ [cf. Eq. (14)] into Eq. (37), we finally obtain the expression for the order parameter $\langle \mathcal{O} \rangle$ near the critical temperature (T_c) as

$$\langle \mathcal{O} \rangle = \beta T_c^2 \sqrt{1 - \frac{T}{T_c}}, \tag{38}$$

where the coefficient β is given by

$$\beta = \frac{16\sqrt{2}\pi^2}{9\sqrt{(\mathcal{A}_1 + \mathcal{A}_2)}}. \tag{39}$$

In the following table (Table II), we have provided both analytic and numerical [62] values for the coefficient β corresponding to different values of the Born-Infeld parameter (b).

Here (from Table II), one can note that both the values that are obtained through different approaches are in the same order. The difference that is caused is mainly due to the perturbative technique itself, where we have dropped higher-order terms in the coupling parameter (b). Similar features have also been found earlier [51]. However, the trend is unique; i.e., β increases as we increase the value of coupling b (see also Fig. 1). Indeed, it would be interesting to carry out the analysis taking into account higher-order terms in the coupling b, which is expected to reduce the disparity between the analytic and numerical results.

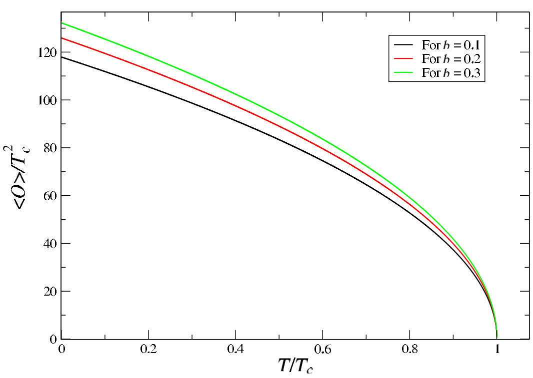

FIG. 1 (color online). Plot of $\langle \mathcal{O} \rangle / T_c^2$ with T/T_c for different values of b.

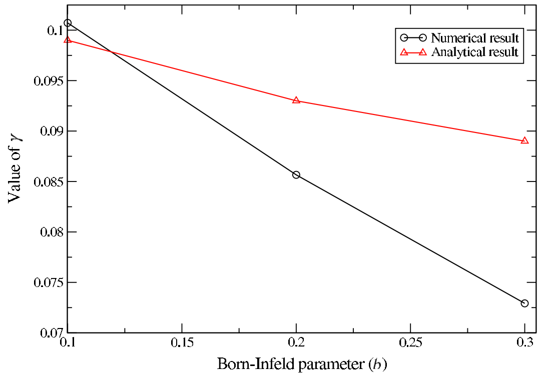

FIG. 2 (color online). The deviation Plot of the coefficient of T_c (γ) with the BI parameter (b).

V. CONCLUSIONS

In this paper, we have considered a holographic model of a superconductor based on fundamental principles of AdS/CFT duality. Among several models of holographic superconductors in the AdS black hole background, we have taken into account a model in which a nonlinear Born-Infeld Lagrangian is included in the matter action. The main purpose for considering the BI theory is that it corresponds to the higher-derivative corrections of the gauge fields in the usual Abelian theory that effectively describes the *low-energy* behavior of the string theory. In this sense, it may be considered as the generalized version of the Abelian model. These corrections must have nontrivial influences on the physical properties of the system.

The aim of the present article is to study the effects of these higher-derivative corrections on the holographic s-wave condensate analytically. In this paper, we have been able to extend the so-called Sturm-Liouville (SL) method for this nonlinear model. This method was first introduced in Ref. [47] in the context of usual Maxwell theory. From our analysis it is indeed evident that extending such a method for the nonlinear model creates difficulties in the analysis. However, we have been able to construct an analytic technique based on this SL method in order to analyze the properties of this holographic superconductor subjected to a nontrivial boundary condition. On top of it, our approach reveals the fact that the solutions of the field equations are highly nontrivial and are not even exactly solvable. The analytic method presented here provides a smooth platform to deal with this difficulty.

The novelty of the present paper is that we have analytically studied the effects of the BI coupling parameter b on the critical temperature and the condensation operator near the critical point. It is observed that the above physical quantities are indeed affected due to the higher-derivative corrections. The results thus obtained from our calculations can be summarized qualitatively as follows:

(i) The critical temperature (T_c) increases as we decrease the value of b, indicating the onset of a harder condensation (Table I).

(ii) The value of the order parameter increases with the increase of b (Table II).

The point that must be stressed at this stage of discussion is that the analytic approach is always preferable to the numerical approach. This is due to the fact that the numerical results become less reliable when the temperature T approaches zero [5,47]. In this temperature limit, the numerical solutions to the nonlinear field equations become much more difficult, and therefore the determination of the nature of the condensate becomes practically very arduous unless analytic methods are taken into account. Therefore, the analytic method is always more reliable while performing computations as $T \longrightarrow 0$.

From Fig. 2 one may note that the results that are obtained using the SL method are not exactly identical to those obtained by numerical techniques. The deviation of the analytic values from the numerical ones (Tables I and II) is not unusual [51], considering the difference in the two approaches (analytic and numerical). Contrary to the numerical approach, in the analytic method we have taken into account only the leading-order terms in the coupling b. Certainly, there is a great amount of approximation involved which is absent in the numerical technique.

The difference between the two approaches in fact motivates us to enquire into a more general analytic approach, in which the above disparity may be reduced and the agreement eventually becomes more close. Apart from this, there are other possibilities which we must emphasize in order to obtain enriched physics from the theoretical point of view. These may be stated as follows:

(i) It would be very interesting to repeat the above analysis in the presence of backreaction.

(ii) With the mathematical technique presented here, one can perform the analysis in higher dimensions.

(iii) Apart from the BI electrodynamics, one can also analyze the problem considering any other nonlinear theory (power Maxwell action, Hoffman-Infeld theory, logarithmic electrodynamics, etc.) existing in the literature.

As a final remark, we would like to mention that the numerical results obtained in the existing literature have always been substantiated by analytic results. However, one may confirm the validity of the analytic results obtained by the Sturm-Liouville (SL) method (without referring to the numerical results) by comparing them with the results obtained from an alternative analytic technique which is known as the matching method [48].

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APPENDIX: DERIVATION OF EQUATION (31)

The *lhs* of Eq. (31) may be written as

$$\frac{d}{dz}(e^{3b\lambda^2 z^4 \xi'^2(z)/2}\chi'(z)) = e^{3b\lambda^2 z^4 \xi'^2(z)/2}[\chi''(z) + 6b\lambda^2 z^3 \xi'^2(z)\chi'(z) + 3b\lambda^2 z^4 \xi'(z)\xi''(z)\chi'(z)]. \quad \text{(A1)}$$

The last term on the rhs of Eq. (A1) can be rewritten as

$$\begin{aligned} 3b\lambda^2 z^4 \xi'(z)\xi''(z)\chi'(z) &= 3b\lambda^2 z^4 \xi'(z)\left(\frac{\phi_0''(z)}{\lambda r_+}\right)\chi'(z) \\ &= \frac{-6b^2\lambda z^7 \xi'(z)}{r_+^3}\phi_0'^3(z)\chi'(z) \\ &= -6b^2\lambda^4 z^7 \xi'^4(z)\chi'(z) \approx 0, \end{aligned} \quad \text{(A2)}$$

where we have used Eq. (13).

Therefore, Eq. (A1) becomes

$$\frac{d}{dz}(e^{3b\lambda^2 z^4 \xi'^2(z)/2}\chi'(z)) = e^{3b\lambda^2 z^4 \xi'^2(z)/2}[\chi''(z) + 6b\lambda^2 z^3 \xi'^2(z)\chi'(z)]. \quad \text{(A3)}$$

The *rhs* of Eq. (31) may be written as

$$\begin{aligned} & e^{3b\lambda^2 z^4 \xi'^2(z)/2}\frac{\lambda \mathcal{F}^2(z) z^2 \xi(z)}{(1-z^3)}\left(1 - \frac{3b\lambda^2 z^4 \xi'^2(z)}{2}\right) \\ &\approx \left(1 + \frac{3b\lambda^2 z^4 \xi'^2(z)}{2}\right)\frac{\lambda \mathcal{F}^2(z) z^2 \xi(z)}{(1-z^3)}\left(1 - \frac{3b\lambda^2 z^4 \xi'^2(z)}{2}\right) \\ &\approx \frac{\lambda \mathcal{F}^2(z) z^2 \xi(z)}{(1-z^3)}. \end{aligned} \quad \text{(A4)}$$

Combining Eqs. (A3) and (A4), we obtain the desired form of Eq. (31).

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Holographic s-wave condensation and Meissner-like effect in Gauss–Bonnet gravity with various non-linear corrections

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In this paper we have studied the onset of holographic s-wave condensate in the $(4+1)$ dimensional planar Gauss–Bonnet-AdS black hole background with several non-linear corrections to the gauge field. In the probe limit, performing explicit analytic computations, with and without magnetic field, we found that these higher order corrections indeed affect various quantities characterizing the holographic superconductors. Also, performing a comparative study of the two non-linear electrodynamics it has been shown that the exponential electrodynamics has stronger effects on the formation of the scalar hair. We observe that our results agree well with those obtained numerically (Zhao et al., 2013).

1. Introduction and motivations

The AdS/CFT correspondence [1,2] has proven to be useful in describing several aspects of strongly coupled field theories from their weakly coupled dual gravity theories which lie in one higher dimension.[1] Over the past several years this correspondence has been extensively used to explore certain field theoretic phenomena where conventional perturbation methods fail to give consistent results [7].

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[1] For excellent reviews on this correspondence see [3–6].

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Among many sectors where this correspondence has been applied successfully, strongly interacting condensed matter theory has been central to the discussion where conventional perturbation techniques appear to be unfaithful. Using this holographic conjecture a holographic model of high T_c superconductors has been proposed [8,9]. Further study of these phenomenological models reveal many of their interesting features which resemble that of the conventional superconductors [10].

Gravity theories in higher dimensions[2] (greater than four) have earned repeated attentions in the past several decades with the advent of string theory. The primary motivation comes from the fact that the consistent description of string theory requires the inclusion of higher dimensional space–time. Certain aspects of string theory are well described by associating gravity theories with it. The effect of string theory on gravity may be studied by considering a low-energy effective action which describes classical gravity [14]. This effective action must contain higher curvature terms and are needed to be ghost free [15]. The Lovelock action is found to be consistent with these criteria [16].

Along with the conventional Maxwell electrodynamic theory non-linear electrodynamic theories (NED), which correspond to the higher derivative corrections to the Abelian gauge fields, have also become interesting topics of research for the past several decades. The primary motivation for introducing non-linear electrodynamic theory was to remove divergences in the self-energy of point-like charged particles [17]. However, they have earned renewed attention over past several years since these theories naturally arise in the low-energy limit of the heterotic string theory [18–20].

Besides the conventional Born–Infeld non-linear electrodynamics (BINE) [17], two new types of NEDs have been proposed recently, namely the exponential non-linear electrodynamics (ENE) and the logarithmic non-linear electrodynamics (LNE), in the context of static charged asymptotic black holes [21,22]. In fact, the matter actions with ENE and LNE yield the higher derivative corrections to the usual Maxwell action. On the other hand these NEDs possess many unique properties which are quite different from the Maxwell electrodynamics. For example, while solutions with LNE completely remove divergences in the electric fields at $r = 0$, these divergences still remain in the solutions with ENE. But these divergences are much weaker than the usual Maxwell case [21,22]. Also, compared with Maxwell theory, solutions with LNE and ENE have different temperatures and electric potentials [21, 22]. Another novel property of these non-linear theories is that, their asymptotic black hole solutions are the same as that of a Reissner–Nordström black hole [22]. On top of that, these types of non-linear theories retain some interesting properties (alike BINE) such as, absence of shock waves, birefringence etc. [21,22]. One further advantage of studying ENE and LNE over the Maxwell theory is that they provide an enriched platform to investigate generalized versions of NEDs in a systematic manner so as to reveal some general features of the effects of higher derivative corrections to the gauge fields in the theory concerned.

At this point of discussion it must be stressed that while gravity theories with NEDs give rise to many interesting gravity solutions which in many respects are different from the solutions with usual Maxwell electrodynamics [23–36], these are also widely discussed in the context of gauge/string duality, specifically in the holographic study of condensed matter phenomena with holographic superconductors as specific examples. Holographic superconductors with non-linear electrodynamics [37–52] have been investigated alongside those with Maxwell electrodynamics [10]. These studies show that the non-linearity in the theory indeed modifies the behaviours of the holographic condensates in non-trivial manners which cannot be observed in the conventional holographic superconductors with Maxwell electrodynamics. In other words the higher derivative corrections to the Abelian gauge fields are manifested as effects on certain properties of the dual holographic models. This observation motivates us to study models of holographic superconductors with ENE and LNE and look for the modifications they make on certain properties of the models compared to the Maxwell case. This also encourages us to make a comparative study between holographic models with different NEDs regarding their effects on the condensation formation. In this regard consideration of holographic models with ENE and LNE is another motivation of the present paper.

2 For good reviews on gravity theories in higher dimensions see [11–13].

Along with the study of holographic superconductors in the framework of conventional Einstein gravity several attempts have been made in order to study these objects in the presence of higher curvature corrections to the Einstein gravity in higher dimensions [39,42,45,51–59]. These curvature corrections also modify some of the properties of the holographic superconductors, such as (i) the observed constancy of the ratio of the frequency gap of the real part of the conductivity to the critical temperature of the superconductor [60] breaks down for Gauss–Bonnet superconductors [53,55–57], (ii) in certain generalized cases of different values of the Gauss–Bonnet correction changes the order of the phase transition [57], (iii) the ratio of the shear viscosity to the entropy density ($\eta/s \geq 1/4\pi$) in CFT dual to the Einstein–Gauss–Bonnet gravity changes significantly with the Gauss–Bonnet coupling [61,62].

Apart from these, the effects of external magnetic fields on the holographic superconductors with or without these nontrivial non-linear corrections have been studied. These studies reveal several interesting properties of these superconductors which resemble certain properties of conventional superconductors, such as the Meissner effect, vortex and droplet solutions etc. [43,47,63–70].

It must be emphasized that holographic superconductors with several NEDs (BINE, ENE, LNE) has been studied in Ref. [49] in the planar Schwarzschild-AdS black hole background without taking into account higher curvature corrections to the Einstein gravity. Also, in Ref. [52] a holographic model with BINE in the Gauss–Bonnett gravity has been considered. Surprisingly most of the computations have been performed using numerical methods. Therefore, it would be nice to perform analytic analysis of holographic superconductor models with non-linear electrodynamic fields (ENE and LNE) in the Gauss–Bonnet black hole background.

Considering all the above mentioned facts, in this paper we have made an extensive analytic investigation of the holographic model of superconductors mentioned at the end of the previous paragraph in the presence of an external magnetic field. The primary motivations of our study may be summarized as follows: (i) investigating the effects of higher curvature as well as higher derivative corrections on the properties of holographic superconductors, (ii) exploring the effects of an external magnetic field on the holographic condensates and determining how the non-linear corrections modify the critical value of the magnetic field, (iii) making a comparison between the Maxwell theory and the non-linear electrodynamic theories regarding their effects on the formation of the scalar condensates, (iv) comparing the non-linear electrodynamic theories in an attempt to see which one has stronger effects on the formation of the scalar condensates.

The present paper has been organized as follows. In Section 2, the basic setup for s-wave holographic superconductor with two different non-linear electrodynamics (ENE and LNE) in the planar $(4+1)$-dimensional Gauss–Bonnet AdS black hole background is given. In Section 3, we have calculated various properties of the s-wave holographic superconductor with exponential electrodynamics (ENE) which include the critical temperatures for condensation and the expectation values of the condensation operator in the absence of external magnetic field. In Section 4, we have discussed the effects of an external static magnetic field on this holographic superconductor and calculated the critical magnetic field for condensation. We have drawn our conclusions and discussed some of the future scopes in Section 5. Finally, in the Appendix we have given the expressions of the aforementioned quantities for the s-wave holographic superconductor with logarithmic electrodynamics (LNE) and derived a necessary mathematical relation.

2. Basic set up

The effective action for the higher curvature Lovelock gravity in an arbitrary dimension d may be written as [16],

$$\mathcal{S}_{grav} = \frac{1}{16\pi G_d} \int d^d x \sqrt{-g} \sum_{i=0}^{[d/2]} \alpha_i \mathcal{L}_i \tag{1}$$

where, α_i is an arbitrary constant, $\mathcal{L}_i$ is the *Euler density* of a $2i$ dimensional manifold and G_d is the Gravitational constant in d-dimensions. In our subsequent analysis we shall consider the coordinate system where $G_d = \hbar = k_B = c = 1$.

In this paper we shall be mainly concerned with the (4 + 1)-dimensional Einstein–Gauss–Bonnet gravity in anti-de Sitter (AdS) space. Thus, the effective action (1) can be written as,

$$\begin{aligned} \mathcal{S}_{grav} &= \frac{1}{16\pi}\int d^5x\sqrt{-g}\left(\alpha_0\mathcal{L}_0 + \alpha_1\mathcal{L}_1 + \alpha_2\mathcal{L}_2\right) \\ &= \frac{1}{16\pi}\int d^5x\sqrt{-g}\left(-2\Lambda + \mathcal{R} + \alpha\mathcal{L}_2\right) \end{aligned} \tag{2}$$

where Λ is the cosmological constant given by $-6/l^2$, l being the AdS length, $\alpha_2 \equiv \alpha$ is the Gauss–Bonnet coefficient, $\mathcal{L}_1 = \mathcal{R}$ is the usual Einstein–Hilbert Lagrangian and $\mathcal{L}_2 = R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + \mathcal{R}^2$ is the Gauss–Bonnet Lagrangian.

The Ricci flat solution for the action (2) is given by [53,71,72]

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(dx^2 + dy^2 + dz^2) \tag{3}$$

where the metric function is[3] [53,71,72],

$$f(r) = \frac{r^2}{2a}\left(1 - \sqrt{1 - 4a\left(1 - \frac{M}{r^4}\right)}\right). \tag{4}$$

In (4), M is the mass of the black hole which may be expressed in terms of the horizon radius (r_+) as, $r_+ = M^{1/4}$ [42,53,56]; the parameter a is related to the coefficient α as, $a = 2\alpha$. It is to be noted that, in order to avoid naked singularity we must have $a \leq 1/4$ [42,53], whereas, considering the causality of dual field theory on the boundary the lower bound of a is found to be $a \geq -7/36$ [73]. Also, in the asymptotic infinity ($r \to \infty$) we may write the metric function (4) as,

$$f(r) \sim \frac{r^2}{2a}\left(1 - \sqrt{1-4a}\right). \tag{5}$$

Thus, the effective AdS radius can be defined as [53],

$$L_{eff}^2 = \frac{2a}{1 - \sqrt{1-4a}}. \tag{6}$$

Note that, in the limit $a \to 1/4$, $L_{eff}^2 = 0.5$. This limit is known as the Chern–Simons limit [74].

The Hawking temperature of the black hole may be obtained by analytic continuation of the metric at the horizon (r_+) and is given by [42,53,56],

$$T = \frac{r_+}{\pi}. \tag{7}$$

In this paper we shall study the s-wave holographic superconductor in the framework of various non-linear electrodynamics in the (4 + 1)-dimensional planar Gauss–Bonnet AdS (GBAdS) black hole background. For this purpose, we shall consider a matter Lagrangian which consists of a charged $U(1)$ gauge field, A_μ, and a charged massive complex scalar field, ψ. Thus, the matter action for the theory may be written as [8,9],

$$\mathcal{S}_{matter} = \int d^5x\sqrt{-g}\left(L(F) - |\nabla_\mu\psi - iA_\mu\psi|^2 - m^2\psi^2\right) \tag{8}$$

where m is the mass of the scalar field. Moreover, we shall carry out all the calculations in the probe limit [10]. In this limit, gravity and matter decouple and the backreaction of the matter fields (scalar field and gauge field) can be suppressed in the neutral AdS black hole background. This is achieved by rescaling the matter action (8) by the charge, q, of the scalar field and then considering the limit $q \to \infty$. The generosity of this approach is that we can simplify the problems without hindering

[3] Without loss of generality, we can choose $l = 1$, which follows from the scaling properties of the equation of motion.

the physical properties of the system. The term $L(F)$ in (8) corresponds to the Lagrangian for the non-linear electrodynamic field. In different non-linear theories the Lagrangian $L(F)$ can take the following forms [21,49]:

$$L(F) = \begin{cases} \frac{1}{4b}\Big(e^{-bF^{\mu\nu}F_{\mu\nu}} - 1\Big), & \text{for ENE} \\ \frac{-2}{b}\ \ln\Big(1 + \frac{1}{8}bF^{\mu\nu}F_{\mu\nu}\Big), & \text{for LNE.} \end{cases} \tag{9}$$

Note that in the limit $b \to 0$, we recover the usual Maxwell term: $L(F)|_{b\to 0} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$.

In order to solve the equations of motion resulting from the variation of the action (8) w.r.to the gauge and scalar fields we shall choose the following ansatz for the two fields concerned [43]:

$$A_\mu = \Big(\phi(r), 0, 0, 0, 0\Big), \tag{10a}$$

$$\psi = \psi(r). \tag{10b}$$

It is to be noted that, the above choices of the fields are justified since it is seen that under the transformations $A_\mu \to A_\mu + \partial_\mu\theta$ and $\psi \to \psi e^{i\theta}$ the above equations of motion remain invariant. This demands that the phase of ψ remains constant and we may take ψ to be real without any loss of generality.

With the change of coordinates $z = \frac{r_+}{r}$, where the horizon $(r = r_+)$ is at $z = 1$ and the boundary $(r \to \infty)$ is at $z = 0$, the equations of motion for the scalar field $(\psi(z))$ and the $U(1)$ gauge field (A_μ) may be obtained as [53,56],

$$\psi''(z) + \left(\frac{f'(z)}{f(z)} - \frac{1}{z}\right)\psi'(z) + \frac{\phi^2(z)\psi(z)r_+^2}{z^4 f^2(z)} - \frac{m^2\psi(z)r_+^2}{z^4 f(z)} = 0, \tag{11}$$

$$\left(1 + \frac{4bz^4\phi'^2(z)}{r_+^2}\right)\phi''(z) - \frac{1}{z}\phi'(z) + \frac{8bz^3}{r_+^2}\phi'^3(z) - \frac{2\psi^2(z)\phi(z)r_+^2}{f(z)z^4}e^{-2bz^4\phi'^2(z)/r_+^2} = 0, \quad \text{for ENE} \tag{12}$$

$$\left(1 + \frac{bz^4\phi'^2(z)}{4r_+^2}\right)\phi''(z) - \frac{\phi'(z)}{z} + \frac{5bz^3}{4r_+^2}\phi'^3(z) - \frac{2\psi^2(z)\phi(z)r_+^2}{f(z)z^4}\left(1 - \frac{bz^4\phi'^2(z)}{4r_+^2}\right)^2 = 0, \quad \text{for LNE.} \tag{13}$$

In order to solve the above set of equations we shall choose the following boundary conditions:

(i) At the horizon $(z = 1)$ one must have, for $m^2 = -3$,[4]

$$\phi(1) = 0, \qquad \psi'(1) = \frac{3}{4}\psi(1) \tag{14}$$

(ii) In the asymptotic AdS region $(z \to 0)$ the solutions for the scalar potential and the scalar field may be expressed as,

$$\phi(z) = \mu - \frac{\rho}{r_+^2}z^2, \tag{15a}$$

$$\psi(z) = D_- z^{\lambda_-} + D_+ z^{\lambda_+} \tag{15b}$$

[4] For the rest of the analysis of our paper we choose $m^2 = -3$. This ensures that we are above the Breitenlohner–Freedman bound [75,76].

where $\lambda_{\pm} = 2 \pm \sqrt{4 - 3L_{eff}^2}$ is the conformal dimension of the condensation operator $\mathcal{O}_i$ $(i = 1, 2)$ in the boundary field theory, μ and ρ are identified as the chemical potential and the charge density of the dual field theory, respectively. It is interesting to note that, since we have considered $m^2 < 0$ in our analysis, we are left with the two different condensation operators of different dimensionality corresponding to the choice of quantization of the scalar field ψ in the bulk. We choose $D_- = 0$. Then, according to the AdS/CFT correspondence $D_+ \equiv \langle \mathcal{O}_2 \rangle$, the expectation value of the condensation operator in the dual field theory.

3. *s*-wave condensate and its critical behaviour without magnetic field

In this section we shall analytically derive the critical temperature for condensation, T_c, for the holographic *s*-wave condensate with two types of non-linear electrodynamics mentioned in the previous section. As a next step, we shall determine the normalized condensation operator and the critical exponent associated with the condensation values in the presence of these non-linear theories in the background of $(4 + 1)$-dimensional Gauss–Bonnet AdS black hole. In this way we would be able to demonstrate the effects of the Gauss–Bonnet coupling coefficient (a) as well as non-linear parameter (b) on these condensates.

In order to carry out our analysis we have adopted a well known analytic technique which is known as the matching method [53]. In this method, we first determine the leading order solutions of the equations of motion (11), (12), (13) near the horizon $(1 \geq z > z_m)$ and at the asymptotic infinity $(z_m > z \geq 0)$ and then match these solutions smoothly at the intermediate point, z_m.[5] It is to be noted that the matching method helps us to determine the values of the critical temperature as well as of condensation operator only approximately, in the leading order of the non-linear parameter, b. Moreover, this method provides us a much better understanding of the effects the Gauss–Bonnet coefficient (a) as far as analytic computation is concerned [53].

In this section we shall perform the analysis for the holographic *s*-wave superconductor with exponential electrodynamics only. Since, the analysis for the holographic superconductor with logarithmic electrodynamics closely resemblances to that of the previous one, we shall only present the results corresponding to this model in Appendix A.1.

Let us first consider the solutions of the gauge field, $\phi(z)$, and the scalar field, $\psi(z)$, near the horizon $(z = 1)$. We Taylor expand both $\phi(z)$ and $\psi(z)$ near the horizon as [53],

$$\phi(z) = \phi(1) - \phi'(1)(1 - z) + \frac{1}{2}\phi''(1)(1 - z)^2 + \cdots \tag{16}$$

$$\psi(z) = \psi(1) - \psi'(1)(1 - z) + \frac{1}{2}\psi''(1)(1 - z)^2 + \cdots \tag{17}$$

It is to be noted that, in (16) and (17), we have considered $\phi'(1) < 0$ and $\psi(1) > 0$ in order to make $\phi(z)$ and $\psi(z)$ positive. This can be done without any loss of generality.

Near the horizon, $z = 1$, we may write from (11)

$$\psi''(1) = \left[\frac{1}{z}\psi'(z)\right]_{z=1} - \left[\frac{f'(z)\psi'(z)}{f(z)}\right]_{z=1} - \left[\frac{\phi^2(z)\psi(z)r_+^2}{z^4 f^2(z)}\right]_{z=1} - \left[\frac{3\psi(z){r_+}^2}{z^4 f(z)}\right]_{z=1}. \tag{18}$$

5 In this paper we pursue our analytic investigation in the same spirit as in Refs. [47,53] and match the leading order solutions near the horizon and the boundary at the intermediate point $z_m = 0.5$. It may be stressed that the qualitative features of the analytical approximation does not change for other values of z_m $(0 < z_m \leq 1)$ and differences in the numerical values are not too large [53]. Therefore throughout our analysis we shall choose $z_m = 0.5$ while obtaining numerical values and plotting various quantities. In fact, as mentioned in Ref. [77], this choice of the matching point roughly corresponds to the geometrical mean of the horizon radius and the AdS scale. Interestingly, with this choice of z_m our results are fairly consistent with Ref. [53] for $b = 0$.

Using the L'Hôpital's rule and the values $f'(1) = -4r_+^2, f''(1) = 4{r_+}^2 + 32a{r_+}^2$, we may express (18) in the following form:

$$\psi''(1) = -\frac{5}{4}\psi'(1) + 8a\psi'(1) - \frac{\phi'^2(1)\psi(1)}{16{r_+}^2}. \tag{19}$$

Finally, using the boundary condition (14), the Taylor expansion (17) can be rewritten as,

$$\psi(z) = \frac{1}{4}\psi(1) + \frac{3}{4}\psi(1)z + (1-z)^2\left[-\frac{15}{64} + \frac{3a}{2} - \frac{\phi'^2(1)}{64{r_+}^2}\right]\psi(1). \tag{20}$$

Near the horizon, $z = 1$, from (12) we may write

$$\phi''(1) = \frac{1}{\left(1 + \frac{4b}{r_+^2}\phi'^2(1)\right)}\left[\phi'(1) - \frac{8b}{{r_+}^2}\phi'^3(1) - \frac{\psi^2(1)\phi'(1)}{2}e^{-2b\phi'^2(z)/r_+^2}\right]. \tag{21}$$

In obtaining (21) we have considered that the metric function, $f(z)$, can also be Taylor expanded as in (16), (17).

Substituting (21) in (16) and using (14) we finally obtain,

$$\phi(z) = -\phi'(1)(1-z) + \frac{1}{2}(1-z)^2\left[1 - \frac{8b}{{r_+}^2}\phi'^2(1) - \frac{\psi^2(1)}{2}e^{-2b\phi'^2(z)/r_+^2}\right] \times \frac{\phi'(1)}{\left(1 + \frac{4b}{r_+^2}\phi'^2(1)\right)}. \tag{22}$$

Now, using the method prescribed by the matching technique [53], we match the solutions (20), (22), (15a) and (15b) at the intermediate point $z = z_m$. It is very much evident that the matching of the two asymptotic solutions smoothly at $z = z_m$ requires the following four conditions:

$$\mu - \frac{\rho z_m^2}{r_+^2} = \beta(1-z_m) - \frac{\beta}{2}(1-z_m)^2\left[\frac{1 - 8b\tilde{\beta}^2}{1 + 4b\tilde{\beta}^2} - \frac{\alpha^2}{2}\frac{e^{-2b\tilde{\beta}^2}}{1 + 4b\tilde{\beta}^2}\right] \tag{23}$$

$$-\frac{2\rho z_m}{r_+^2} = -\beta + \beta(1-z_m)\left[\frac{1 - 8b\tilde{\beta}^2}{1 + 4b\tilde{\beta}^2} - \frac{\alpha^2}{2}\frac{e^{-2b\tilde{\beta}^2}}{1 + 4b\tilde{\beta}^2}\right] \tag{24}$$

$$D_+ z_m^{\lambda_+} = \frac{\alpha}{4} + \frac{3\alpha z_m}{4} + \alpha(1-z_m)^2\left[\frac{-15}{64} + \frac{3a}{2} - \frac{\tilde{\beta}^2}{64}\right] \tag{25}$$

$$\lambda_+ D_+ z_m^{\lambda_+} = \frac{3\alpha z_m}{4} - 2\alpha z_m(1-z_m)\left[\frac{-15}{64} + \frac{3a}{2} - \frac{\tilde{\beta}^2}{64}\right] \tag{26}$$

where we have set $\psi(1) = \alpha$, $-\phi'(1) = \beta$ $(\alpha, \beta > 0)$, $\tilde{\beta} = \frac{\beta}{r_+}$ and $D_- = 0$ [cf. (15b)].

From (24), using (7), we obtain,

$$\alpha^2 = \frac{2z_m}{(1-z_m)}e^{2b\tilde{\beta}^2}\left(1 + \frac{4b\tilde{\beta}^2(3-2z_m)}{z_m}\right)\left(\frac{T_c}{T}\right)^3\left(1 - \frac{T^3}{T_c^3}\right). \tag{27}$$

Since, in our entire analysis we intend to keep terms which are only linear in the non-linear parameter, b, (27) can be approximated as,

$$\alpha^2 \approx \frac{2z_m}{(1-z_m)}\left(1 + \frac{6b\tilde{\beta}^2(2-z_m)}{z_m}\right)\left(\frac{T_c}{T}\right)^3\left(1 - \frac{T^3}{T_c^3}\right). \tag{28}$$

Here the quantity T_c may be identified as the critical temperature for condensation and is given by,

$$T_c = \left[\frac{2\rho}{\tilde{\beta}\pi^3}\left(1 - \frac{12b\tilde{\beta}^2(1-z_m)}{z_m}\right)\right]^{\frac{1}{3}}. \tag{29}$$

Now from (25) and (26) we obtain,

$$D_+ = \frac{(3z_m^2 + 5z_m)}{4\,(\lambda_+ + (2-\lambda_+)z_m)}\left(\frac{1}{z_m}\right)^{\lambda_+}\alpha, \tag{30}$$

$$\tilde{\beta} = 8\left[\frac{-15}{64} + \frac{3a}{2} - \frac{(3z_m^2+5z_m)(1+\lambda_+)}{4\,(\lambda_+ + (2-\lambda_+)z_m)\left(1-4z_m+3z_m^2\right)} + \frac{(1+6z_m)}{4\left(1-4z_m+3z_m^2\right)}\right]^{\frac{1}{2}}. \tag{31}$$

Finally, using (7), (28) and (30), near the critical temperature, $T \sim T_c$, we may write the expectation value, $\langle\mathcal{O}_2\rangle$, of the condensation operator in the following form[6]:

$$\frac{\langle\mathcal{O}_2\rangle^{\frac{1}{\lambda_+}}}{T_c} = \left(\frac{\pi}{z_m}\right)\left[\frac{(3z_m^2+5z_m)}{4\,(\lambda_+ + (2-\lambda_+)z_m)}\right]^{\frac{1}{\lambda_+}} \times\left[\frac{6z_m}{(1-z_m)}\left(1+\frac{6b\tilde{\beta}^2(2-z_m)}{z_m}\right)\left(1-\frac{T}{T_c}\right)\right]^{\frac{1}{2\lambda_+}}. \tag{32}$$

In (32) we have normalized $\langle\mathcal{O}_2\rangle$ by the critical temperature, T_c, to obtain a dimensionless quantity [53]. In the similar fashion we can also calculate the critical temperature, T_c, and condensation operator, $\langle\mathcal{O}_2\rangle$, for the holographic superconductors with LNE. The corresponding expressions for the above mentioned quantities are given in Appendix A.1.

In the next section we shall be mainly concerned with the effects of magnetic field on this *s*-wave holographic superconductor with the two different types of non-linear electrodynamics mentioned earlier. But, before that we would like to make some comments on the results obtained so far. These may be put as follows:

(i) From (29) (and (A.1)) it is evident that in order to have a meaningful notion of the critical temperature, T_c, there must have an upper bound to the non-linear coupling parameter, b. The upper bounds corresponding to two non-linear theories are given below:

$$b \le \begin{cases} \dfrac{z_m(\lambda_+ + 2z_m - \lambda_+ z_m)}{12(1+96a\lambda_+ - 6(32a-13)(\lambda_+-1)z_m + 3(32a-5)(\lambda_+-2)z_m^2)}, & \text{for ENE} \\ \dfrac{2z_m(\lambda_+ + 2z_m - \lambda_+ z_m)}{3(1+96a\lambda_+) - 18(32a-13)(\lambda_+-1)z_m + 9(32a-5)(\lambda_+-2)z_m^2}, & \text{for LNE.} \end{cases} \tag{33}$$

Note that, with our choice $z_m = 0.5$ and for fixed values of a, this upper bound is smaller for ENE compared to LNE.

(ii) The critical temperature, T_c, decreases as we increase the values of the non-linear parameter, b (Table 1). This feature is general for the two types of holographic superconductors considered in this paper. It must be remarked that, without any non-linear corrections ($b = 0$) the critical temperature is larger than the above two cases. For example, $T_c = 0.1907\rho^{1/3}$ for $a = 0.2$, $z_m = 0.5$. This suggests the onset of a harder condensation. Another nontrivial and perhaps the most interesting feature of our present analysis is that, for a particular value of the non-linear parameter,

6 Here we have used the relation $(1-t^3) = (1-t)(1+t+t^2)$ for any arbitrary variable t.

Table 1
Numerical values of coefficients of T_c for different values of the parameters a, b.

b	$a = -0.19$		$a = -0.10$	
	ENE	LNE	ENE	LNE
0.0002	0.2113	0.2174	0.2005	0.2081
0.0004	0.2039	0.2165	0.1910	0.2070
0.0006	0.1958	0.2157	0.1805	0.2060
0.0008	0.1871	0.2148	0.1686	0.2049
0.0010	0.1775	0.2139	0.1547	0.2039
0.0012	0.1667	0.2130	0.1377	0.2028
0.0014	0.1542	0.2122	0.1149	0.2016
0.0016	0.1394	0.2113	0.0753	0.2005
0.0018	0.1204	0.2104	–	–
0.0020	0.0923	0.2094	–	–
b	$a = 0.10$		$a = 0.25$	
	ENE	LNE	ENE	LNE
0.0002	0.1834	0.1943	0.1725	0.1861
0.0003	0.1766	0.1936	0.1636	0.1852
0.0004	0.1692	0.1928	0.1537	0.1843
0.0005	0.1610	0.1920	0.1421	0.1834
0.0006	0.1520	0.1913	0.1285	0.1824
0.0007	0.1417	0.1906	0.1110	0.1815
0.0008	0.1296	0.1898	0.0852	0.1805
0.0009	0.1148	0.1890	–	–
0.0010	0.0946	0.1882	–	–

b, the value of the critical temperature, T_c, for the holographic condensate with ENE is smaller than that with LNE (Table 1) showing stronger effects of the former on the condensation. It is also noteworthy that similar feature was obtained numerically by the authors of [49] in the planar Schwarzschild-AdS black hole background.

(iii) The condensation gap for the holographic condensate with non-linear electrodynamics is more than that with Maxwell electrodynamics (Fig. 1). On top of that, holographic superconductors with ENE exhibit larger gap compared with that with LNE. This suggests that the formation of the scalar hair is more difficult for the holographic condensate with ENE [49].

(iv) The Gauss–Bonnet parameter (a) also has important consequences in the formation of the holographic condensate. From Table 1 it is clear that as we increase the value of a the critical temperature for condensation decreases. This means that the increase of a makes the formation of scalar hair difficult. This indeed shows that both a and b has the same kind of influences on the formation of the hair. However, from Fig. 2 we observe that T_c decreases more rapidly with b than with a. This clearly suggests that the Born–Infeld parameter (b) modifies the critical temperature more significantly than the Gauss–Bonnet parameter (a).

(v) The expectation value of the condensation operator, $\langle \mathcal{O}_2 \rangle$, vanishes at the critical point $T = T_c$ and the condensation occurs below the critical temperature, T_c (see the right panel of Fig. 1). Moreover, form (32) (and (A.2)) we observe that $\langle \mathcal{O}_2 \rangle \propto (1 - T/T_c)^{1/2}$ which shows the mean field behaviour of the holographic condensates and signifies that there is indeed a second order phase transition (critical exponent 1/2). This also admires the consistency of our analysis.

4. Effects of external magnetic field with non-linear corrections

In this section we intend to study the effect of an external magnetic field on the holographic superconductors with the non-linear electrodynamics mentioned earlier. But before we present our analysis, we would like to briefly mention the *Meissner-like effect* in the context of holographic superconductors [63] which will be central to our discussion. It is observed that when immersed in an external magnetic field, ordinary superconductors expel magnetic field lines thereby exhibiting perfect diamagnetism when the temperature is lowered through T_c. This is the Meissner effect

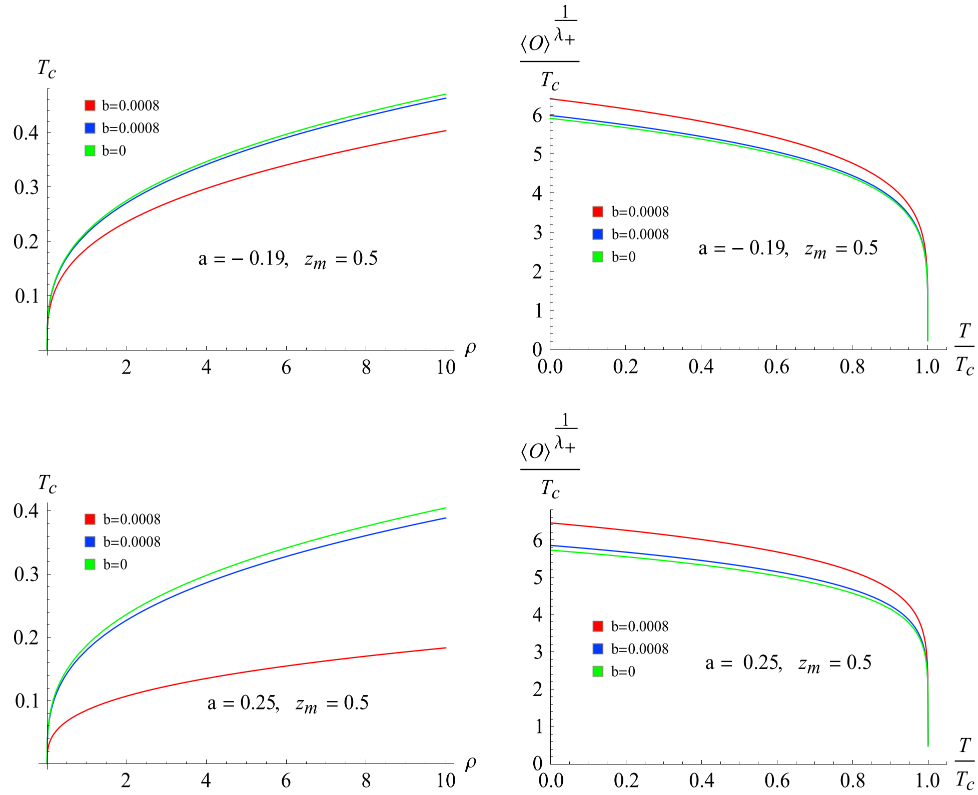

Fig. 1. Plots of critical temperature ($T_c - \rho$) and normalized condensation operator ($\langle \mathcal{O}_2 \rangle^{\frac{1}{\lambda_+}}/T_c - T/T_c$) for different electrodynamic theories. The green, blue and red curves correspond to Maxwell, LNE and ENE, respectively. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

[78]. But for the holographic superconductors in the probe limit we neglect the backreaction of the scalar field on the background geometry. As a result the superconductors are not able to repel the background magnetic field. Instead the scalar condensates adjust themselves such that they only fill a finite strip in the plane which reduces the total magnetic field passing through it. In other words, the effect of the external magnetic field is such that it always tries to reduce the condensate away making the condensation difficult to set in. Considering this apparent similarity with the conventional Meissner effect, this holographic phenomena is referred to as *Meissner-like effect*.

In order to study the effects of magnetic field on the holographic superconductors we add an external static magnetic field in the bulk. According to the gauge/gravity duality, the asymptotic value of the magnetic field in the bulk corresponds to a magnetic field in the boundary field theory, i.e., $B(\mathbf{x}) = F_{xy}(\mathbf{x}, z \to 0)$ [63,65]. Considering the fact that, near the critical magnetic field, B_c, the value of the condensate is small, we may consider the scalar field ψ as a perturbation near B_c. This allows us to adopt the following ansatz for the gauge field and the scalar field [43,47,63,65]:

$$A_\mu = \Big(\phi(z), 0, 0, Bx, 0\Big), \tag{34a}$$

$$\psi = \psi(x, z). \tag{34b}$$

With the help of (8), (34a) and (34b) we may write the equation of motion for the scalar field $\psi(x, z)$ as [43,47,52],

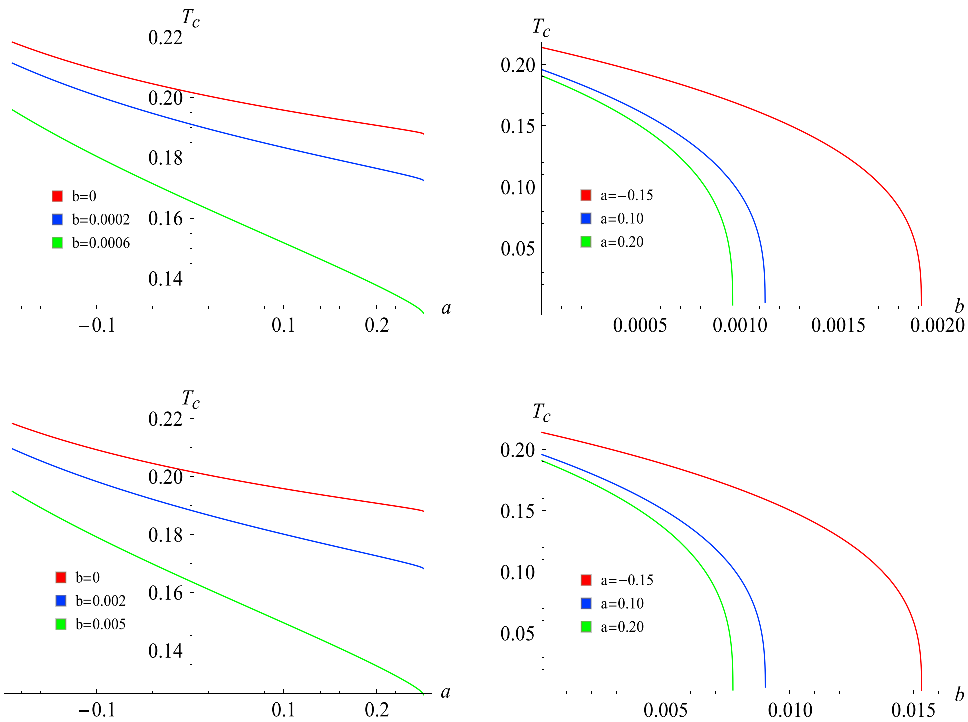

Fig. 2. Left panel: Variation of T_c with a for different values of b (top for ENE, bottom for LNE); Right panel: Variation of T_c with b for different values of a (top for ENE, bottom for LNE). We have chosen $z_m = 0.5$ and $\rho = 1$.

$$\psi''(x,z) - \frac{\psi'(x,z)}{z} + \frac{f'(z)}{f(z)}\psi'(x,z) + \frac{r_+^2\phi^2(z)\psi(x,z)}{z^4 f^2(z)} + \frac{1}{z^2 f(z)}\left(\partial_x^2\psi - B^2x^2\psi\right) + \frac{3r_+^2\psi(x,z)}{z^4 f(z)} = 0 \tag{35}$$

In order to solve (35) we shall use the method of separation of variables [63,65]. Let us consider the solution of the following form:

$$\psi(x,z) = X(x)R(z). \tag{36}$$

As a next step, we shall substitute (36) into (35). This yields the following equation which is separable in the two variables, x and z.

$$z^2 f(z)\left[\frac{R''(z)}{R(z)} + \frac{R'(z)}{R(z)}\left(\frac{f'(z)}{f(z)} - \frac{1}{z}\right) + \frac{r_+^2\phi^2(z)}{z^4 f^2(z)} + \frac{3r_+^2}{z^4 f(z)}\right] - \left[-\frac{X''(x)}{X(x)} + B^2x^2\right] = 0. \tag{37}$$

It is interesting to note that, the x dependent part of (37) is localized in one dimension. Moreover, this is exactly solvable since it maps the quantum harmonic oscillator (QHO). This may be identified as the Schrödinger equation for the corresponding QHO with a frequency determined by B [43,47,63,65],

$$-X''(x) + B^2x^2X(x) = C_n BX(x) \tag{38}$$

where $C_n = 2n + 1$ ($n = integer$). Since the most stable solution corresponds to $n = 0$ [43,47,63], the z dependent part of (37) may be expressed as

$$R''(z) + \left(\frac{f'(z)}{f(z)} - \frac{1}{z}\right)R'(z) + \frac{r_+^2\phi^2(z)R(z)}{z^4 f^2(z)} + \frac{3r_+^2 R(z)}{z^4 f(z)} = \frac{BR(z)}{z^2 f(z)}. \tag{39}$$

Now at the horizon, $z = 1$, using (14) and (39), we may write the following equation:

$$R'(1) = \left(\frac{3}{4} - \frac{B}{4r_+^2}\right) R(1). \tag{40}$$

On the other hand, at the asymptotic infinity, $z \to 0$, the solution of (39) can be written as

$$R(z) = D_- z^{\lambda_-} + D_+ z^{\lambda_+}. \tag{41}$$

It is to be noted that in our analysis we shall choose $D_- = 0$ as was done in Section 3.
Near the horizon, $z = 1$, Taylor expansion of $R(z)$ gives

$$R(z) = R(1) - R'(1)(1-z) + \frac{1}{2}R''(1)(1-z)^2 + \cdots \tag{42}$$

where we have considered $R'(1) < 0$ without loss of generality.
Now calculating $R''(1)$ from (39) and using (40) we may write from (42)

$$\begin{aligned} R(z) &= \frac{1}{4}R(1) + \frac{3z}{4}R(1) + (1-z)\frac{B}{4r_+^2}R(1) \\ &\quad + \frac{1}{2}(1-z)^2\left[3a - \frac{15}{32} + (1-16a)\frac{B}{16r_+^2} + \frac{B^2}{32r_+^4} - \frac{\phi'^2(1)}{32r_+^2}\right]R(1) \end{aligned} \tag{43}$$

where in the intermediate step we have used the Leibniz rule [cf. (18)].
Finally, matching the solutions (41) and (43) at the intermediate point $z = z_m$ and performing some simple algebraic steps as in Section 3 we arrive at the following equation in B:

$$\begin{aligned} & B^2 + 2Br_+^2\left[\frac{8\,(\lambda_+ - (\lambda_+ - 1)z_m)}{(1-z_m)(\lambda_+ - \lambda_+ z_m + 2z_m)} + (1-16a)\right] \\ & \quad + \left[\frac{(1+3z_m)\lambda_+ - 3z_m}{2(1-z_m)(\lambda_+ - \lambda_+ z_m + 2z_m)} + \left(3a - \frac{15}{32} - \frac{\phi'^2(1)}{32r_+^2}\right)\right]32r_+^4 = 0. \end{aligned} \tag{44}$$

Eq. (44) is quadratic in B and its solution is found to be of the following form [52]:

$$\begin{aligned} B &= r_+^2\left[\left(\frac{8\,(\lambda_+ - (\lambda_+ - 1)z_m)}{(1-z_m)(\lambda_+ - \lambda_+ z_m + 2z_m)} + (1-16a)\right)^2 - \left(\frac{16\,[(1+3z_m)\lambda_+ - 3z_m]}{(1-z_m)(\lambda_+ - \lambda_+ z_m + 2z_m)}\right.\right. \\ &\quad \left.\left. + \left(96a - 15 - \frac{\phi'^2(1)}{r_+^2}\right)\right)\right]^{\frac{1}{2}} - r_+^2\left(\frac{8\,(\lambda_+ - (\lambda_+ - 1)z_m)}{(1-z_m)(\lambda_+ - \lambda_+ z_m + 2z_m)} + (1-16a)\right). \end{aligned} \tag{45}$$

We are interested in determining the critical value of the magnetic field strength, B_c, above which the superconducting phase disappears. In this regard, we would like to consider the case for which $B \sim B_c$. Interestingly, in this case the condensation becomes vanishingly small and we can neglect terms that are quadratic in ψ. Thus, the equation of motion corresponding to the gauge field (Eq. (12)), ϕ, may be written as

$$\left(1 + \frac{4bz^4\phi'^2(z)}{r_+^2}\right)\phi''(z) - \frac{1}{z}\phi'(z) + \frac{8bz^3}{r_+^2}\phi'^3(z) = 0. \tag{46}$$

In order to solve the above equation we shall consider a perturbative solution of the following form:

$$\phi(z) = \phi_0(z) + \frac{b}{r_+^2}\phi_1(z) + \cdots \tag{47}$$

where $\phi_0(z)$ is the solution of $\phi(z)$ for $b = 0$ and so on. After some algebraic calculations we may write the solution of (47) as[7]

$$\phi(z) = \frac{\rho}{r_+^2}(1-z^2)\left[1 + \frac{b}{r_+^2} - \frac{2b\rho^2}{r_+^6}(1+z^4)(1+z^2)\right]. \tag{48}$$

At this point of discussion, it must be stressed that we have considered terms which are linear in the non-linear parameter b.

At the asymptotic boundary of the AdS space, $z = 0$, the solution (48) can be approximated as

$$\phi(z) \approx \frac{\rho}{r_+^2}\left[1 + \frac{b}{r_+^2} - \frac{2b\rho^2}{r_+^6}\right] - \frac{\rho}{r_+^2}\left(1 + \frac{b}{r_+^2}\right)z^2. \tag{49}$$

Now, comparing (49) with (15a) we may identify the chemical potential, μ, as

$$\mu = \frac{\rho}{r_+^2}\left[1 + \frac{b}{r_+^2} - \frac{2b\rho^2}{r_+^6}\right]. \tag{50}$$

Near the horizon, $z = 1$, we may write from (46)

$$\phi''(1) = \phi'(1) - \frac{12b}{r_+^2}\phi'^3(1) + \mathcal{O}(b^2). \tag{51}$$

Substituting (51) into (16) and using the boundary condition (14) we may write

$$\phi(z) = -\phi'(1)(1-z) + \frac{1}{2}(1-z)^2\left(\phi'(1) - \frac{12b}{r_+^2}\phi'^3(1)\right). \tag{52}$$

Matching the solutions (52) and (15a) at the intermediate point z_m and using (50) we can find the following relation:

$$(\beta - 2\eta)(2\eta^3 - 6\beta^3 - \eta) = 0 \tag{53}$$

where we have set $-\phi'(1) = \beta$ and $\frac{\rho}{r_+^2} = \eta$. One of the solutions of this quartic equation can be written as

$$\beta = 2\eta$$

which implies

$$\phi'(1) = -\frac{2\rho}{r_+^2}. \tag{54}$$

As a final step, substituting (54) into (45) and using (7) and (29) we obtain the critical value of the magnetic field strength as

$$\frac{B_c}{T_c^2} = \pi^2\left(1 + \frac{12b\tilde{\beta}^2(1-z_m)}{\mathcal{C}^2 z_m}\right)\left[\tilde{\beta}\mathcal{C} - \mathcal{M}\left(\frac{T}{T_c}\right)^3\right]. \tag{55}$$

In a similar manner we can determine the critical value of magnetic strength, B_c, for the holographic superconductor with logarithmic electrodynamics. The expression for B_c in this model is given in Appendix A.1.

In (55) the terms $\mathcal{C}$ and $\mathcal{M}$ can be identified as

$$\mathcal{C} = \left(1 - \frac{\mathcal{A} - \mathcal{M}^2}{\tilde{\beta}^2}x^6\right)^{\frac{1}{2}}$$

[7] See Appendix for the derivation.

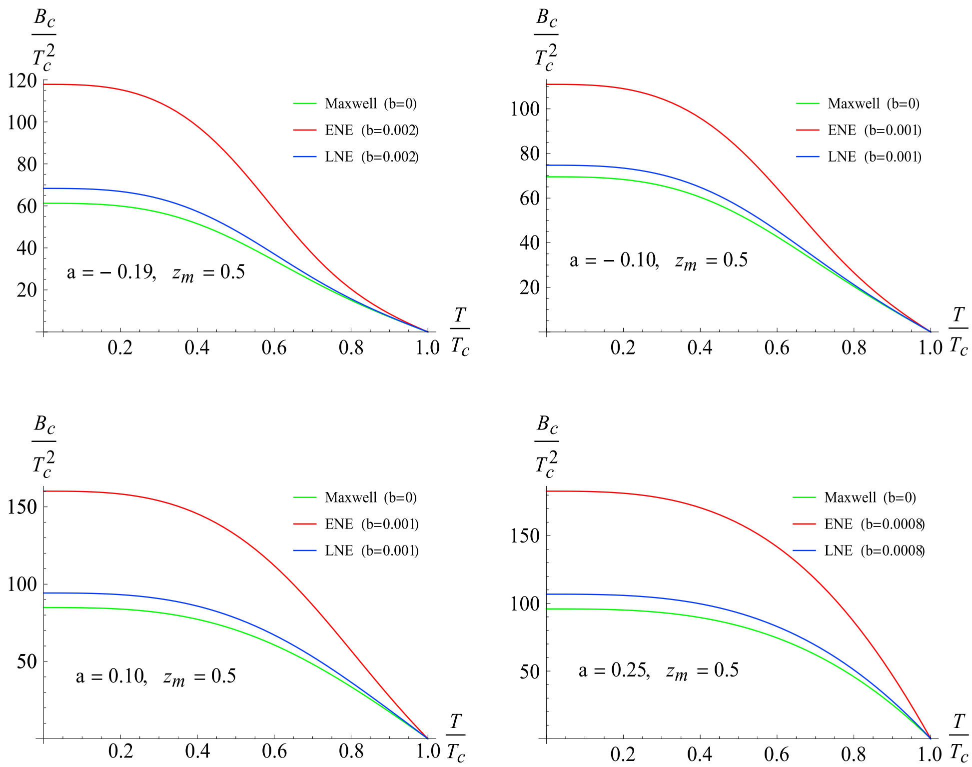

Fig. 3. Plots of $B_c/T_c^2 - T/T_c$ for different values of a and b.

$$\mathcal{M} = 1 - 16a + \frac{8(\lambda_+ - (\lambda_+ - 1)z_m)}{(1 - z_m)(\lambda_+ - \lambda_+ z_m + 2z_m)}$$

where

$$\mathcal{A} = \left[96a - 15 + 16\left(\frac{(1 + 3z_m)\lambda_+ - 3z_m}{(1 - z_m)(\lambda_+ - \lambda_+ z_m + 2z_m)}\right)\right].$$

Here, we must mention that we have normalized B_c by the square of the critical temperature, T_c, such that the critical magnetic field strength, B_c, becomes dimensionless.

In Fig. 3 we have plotted (55) (and (A.3)) as a function of T/T_c. From these plots it is evident that above the critical magnetic field (B_c) the superconductivity is completely destroyed which is also the case for ordinary type II superconductors [78]. From the above analysis we can explain the effects of the Gauss–Bonnet coupling parameter (a) and the non-linear parameter (b) on the holographic condensates. First of all we note that, for fixed values of a the critical magnetic field (B_c) increases with b. Secondly, the critical magnetic field corresponding to the Maxwell case ($b = 0$) is lower than those for ENE and LNE. This indicates that the critical magnetic field strength is higher in presence of the non-linear corrections than the usual Maxwell case. Moreover, this increment is larger for the holographic condensate with ENE than that with LNE. Finally, if we vary a while keeping b constant similar effects are observed, i.e. the critical field strength increases with the Gauss–Bonnet parameter (a). From the preceding discussion we may infer that both the higher order corrections indeed make the condensation harder to form. Moreover, between the two non-linear electrodynamics, the exponential electrodynamics has stronger effects on the formation of the holographic s-wave condensate namely, the formation of the scalar hair is more difficult for holographic superconductor with ENE. This can be explained by noting that the increase in the critical field strength (B_c) tries to reduce the condensate away making the condensation difficult to set in [43,47,63].

5. Conclusions

In this paper, considering the probe limit, we have studied a holographic model of superconductor in the higher curvature planar Gauss–Bonnet-AdS black hole background. We have also taken into account two different types of non-linear electrodynamics (exponential and logarithmic non-linear electrodynamics) in the matter Lagrangian which may be considered as higher derivative corrections to the gauge fields in the usual Abelian gauge theory. In addition to that, we have made an analytic investigation on the effects of an external magnetic field on these superconductors.

The primary motivations of the present paper is to study the effects of several non-linear corrections to the gravity and matter sectors of the action (that describes the holographic superconductivity) on the holographic *s*-wave condensates both in presence as well as absence of an external magnetic field. Along with this we aim to make a comparative study among the usual Maxwell electrodynamics and the two NEDs considered in the paper (ENE, LNE) regarding their effects on the formation of holographic condensates. Based on purely analytic methods we have successfully addressed these issues. The main results of our analysis can be put as follows:

- Non-linear electrodynamics has stronger effects on the condensates than the usual Maxwell case. The critical temperature for condensation (T_c) decreases as we increase the values of the non-linear parameter (b) as well as the Gauss–Bonnet coupling parameter (a) (Fig. 1, Table 1). Moreover, b modifies the critical temperature more significantly than a (Fig. 2). On the other hand, the normalized order parameter ($\langle\mathcal{O}_2\rangle^{1/\lambda_+}/T_c$) increases with the increase of b and a (Fig. 1). This implies that, in the presence of the higher order corrections the formation of the scalar hairs become difficult.
- The variation of the order parameter with temperature, $\langle\mathcal{O}_2\rangle \propto (1-T/T_c)^{1/2}$, exhibits a mean-field behaviour. Also, the value of the associated critical exponent is 1/2, which further ensures that the holographic condensates indeed undergo a second order phase transition in going from normal to superconducting phase.
- There exists a critical magnetic field B_c above which the superconductivity ceases to exist (Fig. 3). This property is similar to that of ordinary type II superconductors [78]. Also, the critical magnetic field strength (B_c) increases as we increase both b and a. The increasing magnetic field strength tries to reduce the condensate away completely making the condensation difficult to form.
- From Figs. 1, 3 and Table 1 we further observe that for particular parameter values T_c is less in holographic superconductor with ENE than that with LNE whereas, $\langle\mathcal{O}_2\rangle^{1/\lambda_+}/T_c$ and B_c is more in the previous one. These results suggest that the exponential electrodynamics exhibit stronger effects than the logarithmic electrodynamics.

It is interesting to note that similar conclusions were drawn in Ref. [49] where numerical computations were performed in this direction. Our analytic calculations provide further confirmations regarding this issue. However, the novel feature of our present analysis is that we have been able to study the effect of the higher curvature corrections which was not performed explicitly in Ref. [49].

Although, we have been able to explore several issues regarding *s*-wave holographic superconductors with different non-linear corrections, there are several other nontrivial and important aspects which can be explored in the future. These can be written in the following order:

(i) We have performed our entire analysis in the probe limit. It will be interesting to carry out the analysis by considering the back-reaction on the metric.

(ii) We can further extend our analysis considering higher curvature gravity theories beyond Gauss–Bonnet gravity. Also, the study of holographic *p*-wave and *d*-wave superconductors in presence of these various higher order corrections may be taken into account.

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Appendix

A.1. Holographic condensate with LNE

The critical temperature for condensation:

$$T_c = \left[\frac{2\rho}{\tilde{\beta}\pi^3}\left(1 - \frac{3b\tilde{\beta}^2(1-z_m)}{2z_m}\right)\right]^{\frac{1}{3}}. \tag{A.1}$$

Normalized order parameter:

$$\frac{\langle\mathcal{O}_2\rangle^{\frac{1}{\lambda_+}}}{T_c} = \left(\frac{\pi}{z_m}\right)\left[\frac{(3z_m^2+5z_m)}{4\left(\lambda_+ + (2-\lambda_+)z_m\right)}\right]^{\frac{1}{\lambda_+}} \times \left[\frac{6z_m}{(1-z_m)}\left(1+\frac{3b\tilde{\beta}^2(2-z_m)}{4z_m}\right)\left(1-\frac{T}{T_c}\right)\right]^{\frac{1}{2\lambda_+}}. \tag{A.2}$$

Critical magnetic field strength:

$$\frac{B_c}{T_c^2} = \pi^2\left(1+\frac{3b\tilde{\beta}^2(1-z_m)}{2\mathcal{C}^2 z_m}\right)\left[\tilde{\beta}\mathcal{C} - \mathcal{M}\left(\frac{T}{T_c}\right)^3\right]. \tag{A.3}$$

The quantities $\mathcal{C}$, $\mathcal{M}$ and $\mathcal{A}$ are identified in Section 4.

A.2. Solution of gauge field ($\phi(z)$) near B_c

The equation for the gauge field near B_c for the holographic superconductor with ENE is given by

$$\left(1+\frac{4bz^4\phi'^2(z)}{r_+^2}\right)\phi''(z) - \frac{1}{z}\phi'(z) + \frac{8bz^3}{r_+^2}\phi'^3(z) = 0. \tag{A.4}$$

Let us consider the following perturbative solution of (A.4):

$$\phi(z) = \phi_0(z) + \frac{b}{r_+^2}\phi_1(z) + \cdots \tag{A.5}$$

where $\phi_0(z), \phi_1(z), \cdots$ are independent solutions and the numbers in the suffices of $\phi(z)$ indicate the corresponding order of the non-linear parameter (b).

Substituting (A.5) in (A.4) we obtain

$$\left[\phi_0''(z) - \frac{\phi_0'(z)}{z}\right] + \frac{b}{r_+^2}\left[\phi_1''(z) - \frac{\phi_1'(z)}{z} + 4z^4\phi_0''(z)\phi_0'^2(z) + \frac{8z^3}{r_+^2}\phi_0'^3(z)\right] + \mathcal{O}(b^2) = 0. \tag{A.6}$$

Equating the coefficients of b^0 and b^1 from the l.h.s of (A.6) to zero we may write

$$b^0:\ \phi_0''(z) - \frac{\phi_0'(z)}{z} = 0 \tag{A.7a}$$

$$b^1:\ \phi_1''(z) - \frac{\phi_1'(z)}{z} + 4z^4\phi_0''(z)\phi_0'^2(z) + \frac{8z^3}{r_+^2}\phi_0'^3(z) = 0. \tag{A.7b}$$

Now, using the boundary condition (15a) we may write the solution of (A.7a) as

$$\phi_0(z) = \frac{\rho}{r_+^2}(1-z^2). \tag{A.8}$$

Using (A.8) we may simplify (A.7b) as

$$\phi_1''(z) - \frac{\phi_1'(z)}{z} - 96z^6\left(\frac{\rho}{r_+^2}\right)^3 = 0. \tag{A.9}$$

As a next step, using the asymptotic boundary condition (15a), from (A.9) we obtain the solution of $\phi_1(z)$ as

$$\phi_1(z) = \frac{2\rho^3}{r_+^6}(z^8-1) - \frac{\rho}{r_+^2}(z^2-1). \tag{A.10}$$

Substituting (A.8) and (A.10) in (A.5) we finally obtain the solution of the gauge field as

$$\phi(z) = \frac{\rho}{r_+^2}(1-z^2)\left[1+\frac{b}{r_+^2}-\frac{2b\rho^2}{r_+^6}(1+z^4)(1+z^2)\right]. \tag{A.11}$$

The solution of the gauge field for the holographic superconductor with LNE may be obtained by similar procedure and is given below:

$$\phi(z) = \frac{\rho}{r_+^2}(1-z^2)\left[1+\frac{b}{r_+^2}-\frac{b\rho^2}{4r_+^6}(1+z^4)(1+z^2)\right]. \tag{A.12}$$

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Magnetic response of holographic Lifshitz superconductors: Vortex and Droplet solutions

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ABSTRACT

In this paper a holographic model of *s*-wave superconductor with anisotropic Lifshitz scaling has been considered. In the presence of an external magnetic field our holographic model exhibits both vortex and droplet solutions. Based on analytic methods we have shown that the anisotropy has no effect on the vortex and droplet solutions whereas it may affect the condensation. Our vortex solution closely resembles the Ginzburg–Landau theory and a relation between the upper critical magnetic field and superconducting coherence length has been speculated from this comparison. Using Sturm–Liouville method, the effect of anisotropy on the critical parameters in insulator/superconductor phase transitions has been analyzed.

1. Introduction and motivations

The emergence of the AdS/CFT correspondence [1,2] has opened up new directions in dealing with the strongly correlated systems. Since its discovery, this duality has been extensively used in several areas in physics such as, fluid/gravity correspondence, QCD, and many others [3–6]. In addition, its lucidity and wide range of applicability have led physicists to apply this correspondence in order to understand several strongly coupled phenomena of condensed matter physics [7–9]. But in many examples of condensed matter physics it is often observed that the behaviors of the systems are governed by *Lifshitz-like fixed points*. These fixed points are characterized by the anisotropic scaling symmetry

$$t \to \lambda^z t, \qquad x^i \to \lambda x^i \quad (i = 1, 2, \ldots, d). \tag{1}$$

The exponent z is called the "dynamical critical exponent" and it describes the degree of anisotropy between space and time [10]. These are non-Lorentz invariant points and hence the systems are non-relativistic in nature [11].

There have been several attempts to describe these systems holographically using the standard prescriptions of gauge/gravity duality. But due to the nonrelativistic nature of these systems the dual description has been modified and it provides a gravity dual for systems which are realized by nonrelativistic CFTs [12–15]. The gravity dual to Lifshitz fixed points is described by the Lifshitz metric [16][1]:

$$ds^2 = -r^{2z}dt^2 + \frac{dr^2}{r^2} + r^2 dx^i dx_i \tag{2}$$

which respects the scale transformation equation (1) along with an additional scaling $r \to \lambda^{-1} r$. In the limit $z = 1$ it gives the AdS_{d+2} metric. On the other hand exact black hole solutions in the asymptotically Lifshitz space–time have been found [17–20].

Recently, the AdS/CFT duality has been used to understand diverse properties of high T_c superconductors [21–24]. The studies of these holographic models of superconductor have been extended by including several higher derivative corrections to the usual Einstein gravity as well as in the Maxwell gauge sector (see Ref. [25] and references therein). Along with this the response of the holographic superconductors in external magnetic fields has also been studied [25–38]. These studies show interesting vortex and droplet solutions for these models [29–32,34,37]. Very recently promising conclusions have been drawn regarding the effects of various corrections to the Einstein–Maxwell sector on the aforementioned solutions [39,40].

Over the past few years a series of works have been attempted to understand various properties of HS with Lifshitz scaling [11, 41–46]. Very recently the authors of Refs. [47,48] found out interesting effects of anisotropy on the characterizing properties of HS with Lifshitz scaling as well as the effects of external magnetic fields on them. In spite of these attempts several other important issues have been overlooked which we intend to study in the present paper. Thus the motivations of the present analysis may be put forward as follows: (i) It has been confirmed that the anisotropic scaling plays an important role in affecting the behavior of the holographic condensates [47,48]. Thus it would

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[1] We shall set the radius of the AdS space to unity ($L = 1$) in our analysis.

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be interesting to verify whether the vortex and/or droplet solutions are affected by it; and (*ii*) It would be nontrivial to study the effects of anisotropy on the holographic condensates. Also, the response of the critical parameters of phase transition to this anisotropy can be studied.

The paper is organized as follows: In Section 2 we have developed the vortex lattice solution for the *s*-wave superconductors in a Lifshitz black hole background. In Section 3 we have obtained a holographic droplet solution for this superconductor in the Lifshitz soliton background. Finally, in Section 4 we have drawn conclusions and discussed some of the future scopes.

2. Vortex solution

In this paper we shall construct a $(2+1)$-dimensional holographic superconductor with Lifshitz scaling (Eq. (1)). According to the gauge/gravity duality the gravitational dual to this model will be a $(3+1)$-dimensional Lifshitz space–time with the following action [15]:

$$\mathcal{S}=\frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(R-2\Lambda-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{1}{4}e^{b\phi}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\right) \tag{3}$$

The background over which we intend to work is given by the following four dimensional Lifshitz black hole [17,47]:

$$ds^2=-\frac{\beta^{2z}}{u^{2z}}f(u)dt^2+\frac{\beta^2}{u^2}\left(dx^2+dy^2\right)+\frac{du^2}{u^2f(u)} \tag{4}$$

where we have chosen a coordinate $u=\frac{1}{r}$, such that the black hole horizon is at $u=1$ and the boundary $(r\to\infty)$ is at $u=0$, for mathematical simplicity. In Eq. (4)

$$f(u)=1-u^{z+2},\qquad \beta(T)=\left(\frac{4\pi T}{z+2}\right)^{\frac{1}{z}} \tag{5}$$

T being the Hawking temperature of the black hole.

The matter action for our model can be written as [23],[2]

$$\mathcal{S}_M=\int d^4x\sqrt{-g}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-|\mathcal{D}_\mu\psi|^2-m^2|\psi|^2\right) \tag{6}$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$, $\mathcal{D}_\mu=\partial_\mu-iA_\mu$ $(\mu,\nu=t,x,y,u)$ and m is the mass of the scalar field ψ.

The equations of motion for the scalar field, ψ, and the gauge field, A_μ, can be obtained from Eq. (6) as

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\partial^\mu\psi\right)-A_\mu A^\mu\psi-m^2\psi-iA^\mu\partial_\mu\psi-\frac{i}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}A^\mu\psi\right)=0, \tag{7}$$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}F^{\mu\nu}\right)=j^\nu\equiv i\left(\psi^*\partial^\nu\psi-\psi\left(\partial^\nu\psi\right)^*\right)+2A^\nu|\psi|^2. \tag{8}$$

In order to proceed further we shall consider the following ansatz for the gauge field [31]:

$$A_\mu=(A_t,A_x,A_y,0). \tag{9}$$

[2] We are working in the probe limit where gravity and matter decouple and the backreaction of the matter fields (the charged gauge field and the charged massive scalar field) on the background geometry can be neglected. This simplifies the problem without affecting the physical properties of the system.

We shall make further assumption that the solutions are stationary i.e. independent of time t. Using these we may write Eqs. (7), (8) as a set of coupled differential equations given by

$$\left(u^{3-z}\partial_u\frac{f(u)}{u^{z+1}}\partial_u+\frac{A_t^2}{\beta^{2z}f(u)}-\frac{m^2}{u^{2z}}\right)\psi=\frac{-1}{\beta^2u^{2z-2}}\left(\delta^{ij}\mathcal{D}_i\mathcal{D}_j\psi\right) \tag{10a}$$

$$f(u)\beta^2\partial_u\left(u^{z-1}(\partial_uA_t)\right)+u^{z-1}\Delta A_t=\frac{2\beta^2A_t}{u^{3-z}}\psi^2, \tag{10b}$$

where $i,j=x,y$ and $\Delta=\partial_x^2+\partial_y^2$ is the *Laplacian operator*.

In order to solve the above set of equations we shall invoke the following boundary conditions [31]:

(i) At the asymptotic boundary $(u\to0)$, the scalar field ψ behaves as [47]

$$\psi\sim C_1u^{\Delta_-}+C_2u^{\Delta_+} \tag{11}$$

where $\Delta_\pm=\frac{(z+2)\pm\sqrt{(z+2)^2+4m^2}}{2}$ and the coefficients C_1,C_2 are related to the expectation values of the operators dual to ψ with scaling dimension Δ_- and Δ_+ respectively. For our analysis we shall always choose the *mass-squared*, m^2, of the scalar field above its lower bound given by $m_{LB}^2=\frac{-(z+2)^2}{4}$ [47]. With this condition both the modes are normalizable and we may choose either one of them as the expectation value of the dual operator while the other behaves as the source. For the rest of our analysis we shall choose $C_1=0$. Also, ψ is regular at the horizon, $u=1$.

(ii) The asymptotic values of the gauge field A_μ give the chemical potential (μ) and the external magnetic field $(\mathcal{B})$ as

$$\mu=A_t(\vec{x},u\to0),\qquad \mathcal{B}=F_{xy}(\vec{x},u\to0) \tag{12}$$

where $\vec{x}=x,y$. The regularity of the gauge fields demand that $A_t=0$ and A_i is regular everywhere on the horizon.

We shall further regard the external magnetic field as the only tuning parameter of our theory. Following this we shall assume μ and T of the boundary theory to be fixed and change only $\mathcal{B}$. Considering our model of holographic superconductor analogous to ordinary type-II superconductor, there exists an upper critical magnetic field, $\mathcal{B}_{c_2}$, below which the condensation occurs while above the $\mathcal{B}_{c_2}$ superconductivity breaks down.

As a next step, define the deviation parameter ϵ such that [31]

$$\epsilon=\frac{\mathcal{B}_{c_2}-\mathcal{B}}{\mathcal{B}_{c_2}},\quad \epsilon\ll1. \tag{13}$$

Let us expand the scalar field ψ, the gauge field A_μ and the current j_μ as the following power series in ϵ:

$$\psi(\vec{x},u)=\epsilon^{1/2}\psi_1(\vec{x},u)+\epsilon^{3/2}\psi_2(\vec{x},u)+\ldots, \tag{14a}$$

$$A_\mu(\vec{x},u)=A_\mu^{(0)}+\epsilon A_\mu^{(1)}(\vec{x},u)+\ldots, \tag{14b}$$

$$j_\mu(\vec{x},u)=\epsilon j_\mu^{(1)}(\vec{x},u)+\epsilon^2j_\mu^{(2)}(\vec{x},u)+\ldots \tag{14c}$$

From Eqs. (13) and (14) we may infer the following interesting points:

(i) Since we have chosen $\epsilon\ll1$, we are in fact very close to the critical point,

(ii) The positivity of the deviation parameter implies that $\mathcal{B}_{c_2}$ is always greater than the applied magnetic field $\mathcal{B}$. This ensures that there is always a non-trivial scalar condensation in the theory that behaves as the order parameter.

Another important point that must be stressed is that, in Eq. (14b) $A_\mu^{(0)}$ is the solution to the Maxwell's equation in the absence of scalar condensate $(\psi=0)$. For the rest of our analysis we shall choose the following ansatz:

$$A_\mu^{(0)}=\left(A_t^0(u),0,A_y^0(x),0\right). \tag{15}$$

Now matching the coefficients of ϵ^0 on both sides of Eq. (10b) we may obtain,

$$A_t^0 = \mu\left(1 - u^{2-z}\right), \qquad A_x^0 = 0, \qquad A_y^0 = \mathcal{B}_{c_2} x. \tag{16}$$

On the other hand using Eqs. (14a), (16) and using the following ansatz for $\psi_1(\vec{x}, u)$ [31]

$$\psi_1(\vec{x}, u) = e^{ipy}\phi(x, u; p) \tag{17}$$

where p is a constant, we can write Eq. (10a) as,

$$\left(u^{3-z}\partial_u \frac{f(u)}{u^{z+1}}\partial_u + \frac{(A_t^{(0)}(u))^2}{\beta^{2z} f(u)} - \frac{m^2}{u^{2z}}\right)\phi(x, u; p) = \frac{1}{\beta^2 u^{2z-2}}\left[\partial_x^2 + (p - \mathcal{B}_{c_2} x)^2\right]\phi(x, u; p). \tag{18}$$

We may solve Eq. (18) by using the method of separation of variables [31]. In order to do so we shall separate the variable $\phi(x, u; p)$ as follow:

$$\phi(x, u; p) = \alpha_n(u)\gamma_n(x; p) \tag{19}$$

with the separation constant λ_n ($n = 0, 1, 2, \ldots$).

Substituting Eq. (19) into Eq. (18) we may write the equations for $\alpha_n(u)$ and $\gamma(x; p)$ as

$$u^{2-2z} f(u)\alpha_n''(u) - \left[\frac{(z+1)f(u)}{u^{2z-1}} + (z+2)u^{3-z}\right]\alpha_n'(u) - \frac{m^2}{u^{2z}}\alpha(u) + \frac{(A_t^{(0)})^2}{\beta^{2z} f(u)}\alpha(u) = \frac{\lambda_n \mathcal{B}_{c_2}}{\beta^2 u^{2z-2}}\alpha_n(u), \tag{20a}$$

$$\left(\partial_X^2 - \frac{X^2}{4}\right)\gamma_n(x; p) = \frac{\lambda_n}{2}\gamma_n(x; p). \tag{20b}$$

where we have identified $X = \sqrt{2\mathcal{B}_{c_2}}(x - \frac{p}{\mathcal{B}_{c_2}})$. Following Ref. [26] we can write the solutions of Eq. (20b) in terms of *Hermite functions*, H_n, with eigenvalue $\lambda_n = 2n + 1$ as

$$\gamma_n(x; p) = e^{-X^2/4} H_n(X). \tag{21}$$

Note that we have considered λ_n to be an odd integer. Since the Hermite functions decay exponentially with increasing X, which is the natural physical choice, our consideration is well justified [26]. Moreover, $\lambda_n = 1$ corresponds to the only physical solution for our analysis. Thus we shall restrict ourselves to the $n = 0$ case. With this choice Eq. (21) can be written as

$$\gamma_0(x; p) = e^{-X^2/4} \equiv \exp\left[-\frac{\mathcal{B}_{c_2}}{2}\left(x - \frac{p}{\mathcal{B}_{c_2}}\right)^2\right]. \tag{22}$$

From the above analysis it is clear that λ_n is independent of the constant p. Therefore, a linear combination of the solutions $e^{ipy}\alpha_0(u)\gamma_0(x; p)$ with different values of p is also a solution to the EoM for ψ_1. Thus, following this proposition, we obtain

$$\psi_1(\vec{x}, u) = \alpha_0(u)\sum_{l=-\infty}^{\infty} c_l e^{ip_l y}\gamma_0(x; p_l). \tag{23}$$

At this point of discussion it is interesting to note that Eq. (23) is very similar to the expression for the order parameter of the Ginzburg–Landau (G–L) theory of type-II superconductors in the presence of a magnetic field [49]

$$\psi_{G-L} = \sum_l c_l e^{ip_l y}\exp\left[-\frac{(x - x_l)^2}{2\xi^2}\right] \tag{24}$$

where $x_l = \frac{k\Phi_0}{2\pi\mathcal{B}_{c_2}}$, Φ_0 being the *flux quanta* and ξ is the *superconducting coherence length*. Comparing Eq. (24) with Eq. (22) we may obtain the following relation between the critical magnetic field and the coherence length as

$$\mathcal{B}_{c_2} \propto \frac{1}{\xi^2} \tag{25}$$

which is indeed in good agreement with the result of the G–L theory [49].

We may obtain the *vortex lattice solution* by appropriately choosing c_l and p_l. In order to do so we shall assume periodicity both in the x and y directions characterized by two arbitrary parameters a_1 and a_2. The periodicity in the y direction can be expressed as

$$p_l = \frac{2\pi l}{a_1\xi}, \quad l \in \mathbb{Z}. \tag{26}$$

Using Eqs. (25), (26) we may rewrite Eq. (22) for different values of l as

$$\gamma(x, y) = \sum_{l=-\infty}^{\infty} c_l \exp\left(\frac{2\pi i l y}{a_1\xi}\right)\exp\left[-\frac{1}{2\xi^2}\left(x - \frac{2\pi l\xi}{a_1}\right)^2\right] \tag{27}$$

where the coefficient c_l can be chosen as

$$c_l = \exp\left(\frac{-i\pi a_2 l^2}{a_1^2}\right). \tag{28}$$

As a next step, we rewrite Eq. (27) by using the *elliptic theta function*, $\vartheta_3(v, \tau)$,[3] as

$$\psi_1(\vec{x}, u) = \alpha_0(u)\exp\left(\frac{-x^2}{2\xi^2}\right)\vartheta_3(v, \tau). \tag{29}$$

where v and τ can be identified as

$$v = \frac{y - ix}{a_1\xi}, \qquad \tau = \frac{2\pi i - a_2}{a_1^2}. \tag{30}$$

Following Refs. [34,40] and using the pseudo-periodicity of $\vartheta_3(v, \tau)$ we see that the function $\sigma(\vec{x}) \equiv |\exp(\frac{-x^2}{2\xi^2})\vartheta_3(v, \tau)|^2$ represents a vortex lattice in which the fundamental region is spanned by the following two lattice vectors

$$\vec{v}_1 = a_1\xi\partial_y, \qquad \vec{v}_2 = \frac{2\pi\xi}{a_1}\partial_x + \frac{a_2}{a_1}\partial_y. \tag{31}$$

We may put forward the main results of this section as follows:

(i) From Eq. (29) it is observed that the vortex solution does not depend upon the dynamic exponent z. This suggests that, whether the boundary field theory is relativistic or non-relativistic, the vortex structure remains the same. Although it is interesting to note that the exponent z may have non-trivial effects on the condensation of the scalar field as is evident from Eq. (20a).

(ii) Eq. (29) also suggests that the structure of the vortex lattice is indeed controlled by the superconducting coherence length, ξ. Moreover, the solution has a *Gaussian profile* along the x direction. As the coherence length decreases the lattice structure gradually dies out. This behavior is similar to that of ordinary type-II superconductors [49].

[3] $\vartheta(v, \tau) = \sum_{l=-\infty}^{\infty}\exp(2i\pi vl + i\pi\tau l^2)$.

3. Droplet solution

In this section we shall consider the holographic insulator/superconductor phase transition in the asymptotically Lifshitz space–time.[4] Our primary goal will be to extract the droplet solution for the s-wave holographic Lifshitz superconductor. In order to perform our analysis we shall consider a planar *Lifshitz soliton*[5] background in 5-dimensions which can be written as [47,50],

$$ds^2 = -r^2dt^2 + r^2(dx^2 + dy^2) + \frac{dr^2}{r^2 f(r)} + r^{2z} f(r)d\chi^2 \tag{32}$$

where $f(r) = (1 - \frac{1}{r^{z+3}})$ and the spatial direction χ is compactified to a circle and has a periodicity $\chi = \chi + \pi$. The geometry looks like a cigar in the (r, χ) directions. However, it will be more convenient to work in polar coordinates, $x = \rho\sin\theta$, $y = \rho\cos\theta$ [36, 37]. With this choice Eq. (32) can be written as,

$$ds^2 = -r^2dt^2 + r^2(d\rho^2 + \rho^2 d\theta^2) + \frac{dr^2}{r^2 f(r)} + r^{2z} f(r)d\chi^2. \tag{33}$$

We shall consider Maxwell-scalar action in 5 dimensions as the matter action of our theory which is written as [23],[6]

$$\mathcal{S}_M = \int d^5x\sqrt{-g}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - |\mathcal{D}_\mu\psi|^2 - m^2|\psi|^2\right). \tag{34}$$

In the probe limit we shall choose the following ansatz for the gauge field close to the critical point of phase transition ($\mu \sim \mu_c$, $\psi \sim 0$)

$$A = \mu_c dt + \frac{1}{2}\mathcal{B}\rho^2 d\theta \tag{35}$$

where μ is the chemical potential and $\mathcal{B}$ is the constant external magnetic field related to the vector potential.

The equation of motion for the scalar field, ψ, can be derived by varying the action (34) w.r.t. ψ and is given by

$$\partial_r^2 F(t,r) + \left(\frac{f'(r)}{f(r)} + \frac{(z+4)}{r}\right)\partial_r F(t,r) - \frac{\partial_t^2 F(t,r)}{r^4 f(r)} + \frac{2i\mu_c}{r^4 f(r)}\partial_t F(t,r) + \left[\frac{\partial_\chi^2 H(\chi)}{r^{2z+2} f^2(r) H(\chi)} - \frac{m^2}{r^2 f(r)} - \frac{\mathcal{B}^2\rho^2}{4r^4 f(r)} + \frac{\mu_c^2}{r^4 f(r)} + \frac{\partial_\rho(\rho\partial_\rho U(\rho))}{r^4 f(r)U(\rho)\rho}\right]F(t,r) = 0 \tag{36}$$

where we have used Eq. (35) and considered the following ansatz

$$\psi(t,r,\chi,\rho) = F(t,r)H(\chi)U(\rho). \tag{37}$$

Now, applying the method of separation of variables we finally obtain the following three equations:

$$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho U(\rho)) - \frac{1}{4}\mathcal{B}^2\rho^2 U(\rho) = -k^2 U(\rho), \tag{38a}$$

$$\partial_\chi^2 H(\chi) = -\lambda^2 H(\chi), \tag{38b}$$

$$\partial_r^2 F(t,r) + \left(\frac{f'(r)}{f(r)} + \frac{(z+4)}{r}\right)\partial_r F(t,r) - \frac{\partial_t^2 F(t,r)}{r^4 f(r)} + \frac{2i\mu_c}{r^4 f(r)}\partial_t F(t,r) + \frac{1}{r^4 f(r)}\left[\mu_c^2 - m^2r^2 - k^2 - \frac{\lambda^2}{f(r)r^{2z-2}}\right]F(t,r) = 0, \tag{38c}$$

where λ and k are some arbitrary constants.

Eq. (38b) has the solution of the form

$$H(\chi) = \exp(i\lambda\chi) \tag{39}$$

which gives $\lambda = 2n$, $n \in \mathbb{Z}$, owing to the periodicity of $H(\chi)$ mentioned earlier.

Eq. (38a) is similar to the equation of a *harmonic oscillator* with $k^2 = l|\mathcal{B}|$, $l \in \mathbb{Z}^+$. We shall expect that the lowest mode of excitation ($n = 0$, $l = 1$) will be the first to condensate and will give the most stable solution after condensation [36,37].

At this point let us discuss one of the main results of this paper. From Eq. (38a) we observe that it has the following solution

$$U(\rho) = \exp\left(\frac{-|\mathcal{B}|\rho^2}{4}\right). \tag{40}$$

This suggests that for any finite magnetic field, the holographic condensate will be confined to a finite circular region. Moreover, if we increase the magnetic field this region shrinks to its size and for a large value of the magnetic field this essentially becomes a point at the origin with a nonzero condensate. This is precisely the holographic realization of a superconducting droplet.

As a next step, we shall be interested in solving Eq. (38c) in order to determine a relation between the critical parameters (μ_c and $\mathcal{B}$) in this insulator/superconductor phase transition. In order to do so, we shall further define $F(t,r) = e^{-i\omega t}R(r)$. With this definition we may rewrite Eq. (38c) as,

$$R''(u) + \left(\frac{f'(u)}{f(u)} - \frac{z+2}{u}\right)R'(u) + \frac{1}{f(u)}\left(\mu_c^2 - \mathcal{B} - \frac{m^2}{u^2}\right)R(u) = 0 \tag{41}$$

where $u = \frac{1}{r}$ and we have put $\omega = 0$ since we are interested in perturbations which are marginally stable [37]. Here 'prime' denotes derivative w.r.t. u.

We shall choose a trial function $\Lambda(u)$ such that

$$R(u \to 0) \sim \langle\mathcal{O}_{\Delta_+}\rangle u^{\Delta_+}\Lambda(u) \tag{42}$$

where $\Delta_\pm = \frac{(z+3)\pm\sqrt{(z+3)^2+4m^2}}{2}$, $m_{LB}^2 = \frac{-(z+3)^2}{4}$ [47] and $\Lambda(0) = 1$, $\Lambda'(0) = 0$. Note that we have identified C_2 in Eq. (11) as the expectation value of the condensation operator, $\langle\mathcal{O}_{\Delta_+}\rangle$.

Substituting Eq. (42) into Eq. (41) we finally get,

$$(\mathcal{P}(u)\Lambda'(u))' + \mathcal{Q}(u)\Lambda'(u) + \Gamma\mathcal{R}(u)\Lambda(u) = 0 \tag{43}$$

where $\Gamma = (\mu_c^2 - \mathcal{B})$ and

$$\mathcal{P} = (1 - u^{z+3})u^{2\Delta_+ - z - 2} \tag{44a}$$

$$\mathcal{Q} = [\Delta_+(\Delta_+ - 1)(1 - u^{z+3}) - m^2 - \Delta_+(z + 2 + u^{z+3})]u^{2\Delta_+ - z - 4} \tag{44b}$$

$$\mathcal{R} = u^{2\Delta_+ - z - 2}. \tag{44c}$$

Note that, Eq. (43) is indeed a standard *Sturm–Liouville* eigenvalue equation. Thus, we may write the eigenvalue, Γ, by using the following formula [51]

[4] The insulator/superconductor phase transition is realized in the CFT language as a phase transition in which a large enough $U(1)$ chemical potential, μ, overcomes the mass gap related to the scalar field, ψ. This mechanism allows ψ to condensate above a critical value, μ_c. In fact a soliton background, which includes an extra compactified spatial direction, precisely generates this mass gap resembling an insulating phase.

[5] The Lifshitz soliton solution is obtained by performing a double Wick rotation of the Lifshitz black hole solution [47,50].

[6] We shall again work in the probe limit.

Table 1
Variation of Γ for $z = \frac{1}{2}$ ($m^2_{LB} = -3.0625$).

m^2	−3.0	−2.0	−1.0	1.0	2.0	3
Γ	2.41947	5.51156	7.59806	11.1513	12.7798	14.3487

Table 2
Variation of Γ for $z = \frac{3}{2}$ ($m^2_{LB} = -5.06$).

m^2	−4.5	−3.5	−2.5	−1.5	1.5	2.5	3.5	4.5
Γ	4.96028	7.50645	9.59179	11.479	16.5666	18.1496	19.6951	21.2097

Table 3
Variation of Γ for $z = \frac{5}{2}$ ($m^2_{LB} = -7.5625$).

m^2	−7.0	−5.0	−3.0	−1.0	1.0	3.0	5.0	7.0
Γ	5.61026	10.5782	14.4483	17.9369	21.2117	24.3448	27.3750	30.3261

$$\Gamma = \frac{\int_0^1 du(\mathcal{P}(u)(\Lambda'(u))^2 + \mathcal{Q}(u)\Lambda^2(u))}{\int_0^1 du\mathcal{P}(u)\Lambda^2(u)} = \Gamma(\alpha, z, m^2) \qquad (45)$$

where we have chosen $\Lambda(u) = 1 - \alpha u^{\Delta_+}$. Thus we may argue that, unlike the case of usual holographic superconductors [37], the quantity $\Gamma = \mu_c^2 - \mathcal{B}$ depends on the dynamic critical exponent (z). Therefore we may conclude that the anisotropic scaling indeed manipulates the relation between the parameters of the phase transition. In Tables 1–3 below we have shown the non-trivial dependence of Γ on z.

4. Conclusions and future scopes

In this paper we have focused our attention to the study of a holographic model of s-wave superconductor with Lifshitz scaling in the presence of external magnetic field by using the gauge/gravity duality. Working in the probe limit we have constructed vortex and droplet solutions for our holographic model by considering a Lifshitz black hole and a Lifshitz soliton background, respectively. Unlike the AdS/CFT holographic superconductors there is a non-trivial dynamic exponent in the theory which is responsible for an anisotropy between the temporal and the spatial dimensions of the space–time resulting certain noticeable changes of the properties of the superconductor [11,41–48]. Also, due to the non-relativistic nature of the filed theory, the model is governed by the AdS/NRCFT correspondence [12–15].

The primary motivation of the present study is to verify the possibility of vortex and droplet solutions, which are common to the usual holographic superconductors described by the AdS/CFT correspondence [29–32,34,37], for this class of holographic superconductors as well as to consider the effects of anisotropy on these solutions. Based on purely analytic methods we have been able to construct these solutions. Our analysis shows that, although, the anisotropy has no effects on the vortex lattice solutions, it may have a non-trivial effect on the formation of holographic condensates. Also, a close comparison between our results and those of the Ginzburg–Landau theory reveals the fact that the upper critical magnetic field ($\mathcal{B}_{c_2}$) is inversely proportional to the square of the superconducting coherence length (ξ). This allows us to speculate the behavior of $\mathcal{B}_{c_2}$ with temperature although this requires further investigations which is expected to be explored in the future. On the other hand, based on the method of separation of variables, we have been able to model a holographic droplet solution by working in a Lifshitz soliton background and considering insulator/superconductor phase transition. Our analysis reveals that a holographic droplet is indeed formed in the $\rho - \theta$ plane with a non-vanishing condensate. Also, this droplet grows in size until it captures the entire plane when the external magnetic field $\mathcal{B} \to 0$. Interestingly, it is observed that the anisotropy does not affect the droplet solution. On top of that, we have determined a relation between the critical parameters of the phase transition by using the Sturm–Liouville method [51]. Interestingly, this relation is solely controlled by the dynamic exponent (z) which in turn exhibits the effects of anisotropy on the condensate (cf. Eq. (45)).

Although we have performed detail analytic calculations regarding some subtle issues of holographic Lifshitz superconductors, there might have been even more interesting outcomes that need further explorations. Some of these can be listed as follows:

(i) It will be interesting to carry out an analysis to see whether p-wave as well as d-wave holographic Lifshitz superconductors form vortex and/or droplet solutions. In this regard the effects of anisotropy on these solutions may be studied.

(ii) It is observed that the holographic model of superconductor analyzed in this paper is quite similar to the real world high-T_c superconductors. But, this is a phenomenological model where we have chosen the fields and their interactions by hand [24]. We have not provided any microscopic theory which drives our model of superconductivity. It is expected to have a microscopic theory by proper embedding of the model into the string theory.

(iii) Note that there are nontrivial dependencies of the equations of motion on the dynamic exponent z derived from the actions of our model. This encourages us to obtain the free energy and the R-current for our model and study the effect(s) of anisotropy on them. More specifically, it will be important to look for any corrections to the *usual Ginzburg–Landau current* due to the presence of anisotropy. In this regard we may also study the long-wavelength limit of the results thus obtained.

Apart from the points mentioned above there are several other non-trivial issues, such as the study of the effects of dynamical magnetic fields as well as of backreaction[7] [52,53] on the properties of the s-wave Lifshitz superconductor, the effects of various non-linear corrections in the gauge and/or gravity sector on this superconductor and the study of the holographic model considered here in higher dimensions, that we wish to illuminate in the future.

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